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坦克自行火炮发射动力学

芮筱亭　刘怡昕　于海龙　著

科学出版社

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内 容 简 介

坦克自行火炮发射动力学是研究坦克和自行火炮在发射过程中的受力及运动规律，为坦克和自行火炮性能总体设计与试验提供控制其受力与运动的理论与技术。本书从炮、弹、药、引信武器系统的角度，系统研究了从点火到炮弹弹着点全过程的坦克自行火炮动力学理论、计算、试验，解决了坦克自行火炮多刚柔体系统振动特性计算、特征矢量正交性、动力学精确分析和快速计算等难题，发明了自行火炮、舰载武器、机载武器等6种射击精度总体设计方法、光学杠杆与毫米波雷达联合测试技术、坦克自行火炮连射弹丸起始扰动测试技术、2种高速旋转弹丸三自由度角运动弹道环境模拟试验装置、4种自行火炮末制导炮弹故障诊断装置和方法，实现了我国几代人在野外进行大口径武器弹丸起始扰动与纵向运动联合测试的夙愿，揭示了引信早炸机理和末制导炮弹解体的物理本质，填补了高速旋转弹丸引信机构运动测试国内技术空白，为坦克自行火炮等武器射击精度与发射安全性设计和试验提供了理论基础和技术手段，解决了提高自行火炮和舰载武器及机载武器射击精度、消除引信早炸和末制导炮弹解体故障等重大工程问题。

本书对坦克自行火炮研究具有重要参考价值，可作为火炮、弹道、工程力学、机械系统动力学研究与工程技术应用的科技人员、教师和研究生的教材和参考书。

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LAUNCH DYNAMICS OF TANK AND SELF-PROPELLED ARTILLERY

Xiaoting Rui   Yixin Liu   Hailong Yu

Science Press
Beijing

Contents Introduction

Launch dynamics of tank and self-propelled artillery(TSPA)is used to study motion styles of the TSPA and the force acting on it during launch process, to provide theory and technology controlled the force and the motion styles for general performance design and test of TSPA. In this book, from the viewpoint of weapon system including gun, projectile, propellant and fuze, dynamics in whole process from ignition to falling point of projectile for TSPA is studied systematically in theory, computation and test. The difficulties, such as, computation of vibration characteristics, orthogonality of eigenvectors, accurate analysis and fast calculation of dynamics of multi-rigid-flexible-body system, have been solved. Six general design methods of firing precision for TSPA and weapon carried by helicopter and by warship were invented. Test technology combining optical lever with millimetre wave radar, and projectile initial disturbance of TSPA in uninterrupted fire, two test devices simulating ballistics situation of projectile with three degree of freedom in high speed angular motion, four devices and methods diagnosed the failure for terminal guidance projectile were invented. The long-cherished wish of several generation in China that test initial disturbance combined with longitudinal motion of projectile for big caliber weapon at the same time in outdoor was realized, the mechanism of early burst of fuze and the physical essence of disassembly of terminal guidance projectile were opened out. The technology blank tested fuze framework's motion of a projectile with high rotation speed was filled up in China. These provide base in theory and technology for design and test of firing precision and launch safety of TSPA and other weapons. The important engineering problem improving firing precision of self-propelled artillery and weapon carried by helicopter and weapon carried by warship, eliminating early burst of fuze and disassembly failure of terminal guidance projectile were solved.

This book has important value for studying TSPA and can be used as a textbook or reference book for engineers, teachers and graduated students in the specialties of artillery, ballistics, engineering mechanics and mechanics system dynamics.

序　一

我国坦克与自行火炮研制在经历了仿制、仿研阶段后，进入到了能自主研发威力大、精度高、反应速度快的坦克与自行火炮武器系统新阶段。然而，在当前研发过程中，随着对武器系统功能、性能指标要求的不断提高，用于现代化设计的理论和试验数据，远远滞后于自主创新研发阶段的需求，制约了我国坦克与自行火炮研发技术向更高水平发展。

《坦克自行火炮发射动力学》一书，从炮、弹、药、引信武器系统的角度，在理论、计算、试验方面，深入研究，阐述了坦克与自行火炮武器系统从点火到炮弹落点的全过程中坦克与自行火炮的受力与运动规律。在此基础上，构建了提高火炮射击精度、发射安全性设计和试验的理论与技术手段，提出了武器系统发射动力学建模与计算方法，为武器系统发射装置研制提供了理论支撑。

作者及其合作者以创新思路，自主研制出了多种试验装置，如：提高武器系统密集度试验装置，大口径弹丸起始扰动光学杠杆试验装置，弹丸膛内姿态与纵向运动联合测试装置，高速旋转弹丸三自由度角运动模拟试验装置和控制系统，末制导炮弹发射环境下应力测试装置、振动模态测试装置、发动机密封检测装置、高温应力变化测试装置等，提出了振动模态试验方法、弹丸起始扰动试验方法、动力响应试验方法、末制导炮弹预应力试验方法、末制导炮弹强度试验方法等试验方法。应用书中提出的试验装置和方法测得的试验数据，可为武器系统发射装置设计，提高射击精度，防止膛炸、早炸与末制导炮弹解体提供数据支撑。

基于多体系统传递矩阵法的《坦克自行火炮发射动力学》是一部有很高学术价值的专著，书中用系统论的思想构建的坦克与自行火炮发射动力学模型与设计方法，具有重要的理论意义与实用价值。我认为，该书的出版将使我国坦克与自行火炮发射动力学基础理论、计算方法、试验方法跃上新台阶，也将为提高坦克与自行火炮发射装置设计水平和综合性能发挥重要作用。

中国北方车辆研究所教授
中国工程院院士　王哲荣

2010 年 1 月 5 日

王哲荣：三代坦克副总设计师，著名兵器专家。

Foreword One

Development of tank and self-propelled artillery(TSPA) in China experienced imitation and imitation research stage successively. And now it has come into such a new stage, that is, TSPA weapons systems with strong power, high precision and fast response can be designed and manufactured independently. However, in current process of research and development, along with continual increase of demand for the function and performance of weapons systems, the theory and experiment data using for modern design is far laggard than the demand of innovation design and manufacture stage. This situation hampered the development of design and manufacture technology of China in TSPA to higher level.

In this book *Launch Dynamics of Tank and Self-propelled Artillery*, from the viewpoint of weapon systems including gun, projectile, propellant and fuze, by studying deeply on the theory, computation and experiment, the force acted on tanks and self-propelled artillery and the motion style in the whole process from ignition to the falling point of projectile are expatiated. Based on these, the theory and technology to improve the firing precision, design and experiment of launch safety of gun are constructed, methods modeling and computing for launch dynamics of weapon system are put forward. So, the theoretical foundation is provided for development of launch devices of weapon system.

Many experiment equipments were developed by authors and their co-workers based on their innovative ideas, such as, the experiment equipment improving firing dispersion of weapon system, the optical lever device testing the initial disturbance of large-caliber projectile, the device combined testing orientation and longitudinal motion of projectile in gun tube, the device and its control system simulating angular motion of projectile with high rotation speed and three degrees of freedom, device testing stress of terminal guidance projectile in launch situation, device testing vibration modal, device testing engine sealing, device testing transformation of stress under high temperature. At the same time, the methods, such as, method testing vibration mode, method testing initial disturbance for projectile, method of testing dynamic response, method testing pre-stress of terminal guidance projectile, method testing strength of terminal guidance projectile, and so on, were presented. The experiment data get by applying these test devices and methods proposed in this book can be used to design launch device of weapon system, to improve firing precision, to prevent bore burst, early burst and the disassembly of terminal guidance projectile.

Launch Dynamics of Tank and Self-propelled Artillery is a book based on transfer matrix method of multibody system with high academic valuable. The dynamics model and design method of tanks and self-propelled artillery constructed in the thought of system theory in this book have important theoretical and practical significance. I think, because the publication of this book, our country will jump a new level in basic theory, computation method and test methods of launch dynamics of tanks and self-propelled artillery, and will play an important role in improving design level and general performance of tanks and self-propelled artillery.

Professor Zherong Wang
Professor of China North Vehicle Research Institute
Member of Chinese Academy of Engineering
January 5, 2010

序 二

芮筱亭、刘怡昕教授和于海龙博士的专著《坦克自行火炮发射动力学》凝聚了作者在这一领域多年耕耘的学术成果。该书将发射平台与发射装药、弹丸及引信视为一个系统，研究其发射动态响应、射击精度、弹丸及引信故障等多方面问题，理论研究与应用研究紧密结合，科学揭示其内在规律，创造性地解决了诸多理论难题和工程技术难题，在我国该领域研究中具有引领和标志性作用。

从 20 世纪 70 年代初开始，我参与了我国引信历次膛炸、炮口及弹道早炸以及瞎火等各种故障的分析工作。其中相当多的故障现象利用传统的刚体动力学原理难以得到合理解释。例如，某迫击炮在 20 世纪 70 年代中期多次发生的炮口早炸，通过多种试验方法，消耗弹药上百发，才揭示其早炸机理是发火机构动态响应导致击针往复振动所造成的，属于发射动力学问题，但在当时尚缺乏理论手段予以系统地分析与描述。

1995 年，在主管部门支持下，我和谭惠民教授邀请芮筱亭教授及其合作者将火炮、弹丸和引信结合在一起进行发射动力学研究工作，开展跨学科合作研究。自那时起，15 年来我与筱亭一直保持着密切的学术联系，多次邀请他运用发射动力学研究成果运用理论和试验研究揭示引信及末制导炮弹的弹道早炸和膛内解体等故障机理，取得了符合实际并具有说服力的创新性成果。其中有两项研究特别值得称道。

2002 年，某自行迫榴炮引信发生弹道早炸，我委托筱亭教授进行研究。该引信带有气动涡轮保险机构，需将多体动力学与空气动力学相结合来研究。筱亭不辞辛劳，昼夜加班，带领他的团队从发射动力学原理出发，利用高速旋转弹丸三自由度角运动模拟装置以及 NH-1 高速风洞，对该引信外弹道初始段飞行及自转、章动、进动以及气动弹道环境进行了物理仿真，揭示了该引信在出炮口刚解除保险时击针刺雷管导致早炸的机理，在实验室再现了引信弹道早炸现象，并被后续的射击试验所证实，为其后的引信结构改进及参数调整提供了重要依据。这一研究成果已反映在该书的 12.7 节中。

2008 年，受主管部门委托，由我主持某末制导炮弹解体故障分析专家组工作。在此之前已发生多起 3 号装药发射时炮弹解体故障，经分析后采取了改进措施，但其效果只是降低了解体概率而未能根除解体故障。我接手这项工作并仔细观察炮弹解体残骸后，认为炮弹膛内解体原因复杂，应与末制导炮弹在膛内的发射动力学响应特性紧密相关。遂邀请筱亭教授参与故障诊断工作。筱亭和他的团队首先对

末制导炮弹进行了模态试验与分析，指出该弹发动机与弹体联接处的横向振动变形最大，径向定位不足；又专门设计了密封试验、推进剂药柱预应力测试试验和强度试验，结合模态分析结果提出了末制导炮弹膛内解体故障的机理、故障定位和改进意见。以此为主要基础并综合其他因素专家组提出了解决解体故障的治理措施，并被研制单位采纳。经改进设计后的末制导炮弹在其后续的射击试验中未再出现解体现象。这一研究成果反映在该书第 12 章中。

这两个事例生动地表明，筱亭教授及其合作者不仅在坦克自行火炮发射动力学理论研究中有丰富的理论建树，而且在发射动力学试验研究以及理论与试验有机结合，解决火炮、弹药及引信发射故障实际问题中也取得丰富的创新性研究成果，为我国坦克自行火炮及其弹药引信的装备研制和技术发展做出了突出的贡献。

在《坦克自行火炮发射动力学》付诸出版之际，写此序以志念与筱亭的多年合作，并向三位作者致贺。相信这本书的出版将对进一步推动发射动力学研究及其应用起到重要的作用。

北京理工大学教授
北京理工大学首席专家　马宝华

2010 年 4 月 26 日

马宝华：著名引信专家。

Foreword Two

Prof. Xiaoting Rui, Prof. Yixin Liu and Dr. Hailong Yu have been studied deeply in the field of launch dynamics of tank and self-propelled artillery (TSPA) for many years, the book *Launch Dynamics of Tank and Self-propelled Artillery* shows the summarization of authors' these academic achievements. In this book, the launch platform, propellant charge, projectile and fuze are regarded as one system, many problems, such as, the dynamic response of launch, firing precision, failure of projectile and fuze, and so on, are studied. Theoretical research and applied research are combined closely, the inherent law is revealed scientifically, many theoretical and engineering problems have been solved creatively. These academic achievements played the lead and symbolic role in the field of our country.

From the early 70s of last century, I participated in all of the analysis work of fuze failure, included bore burst of fuze, early burst of fuze in muzzle and in trajectory, blind of fuze, and so on. Many failure phenomena cannot be explained with reasons by principle of traditional rigid body dynamics. For example, early burst in muzzle in a mortar ever occurred for several times in the mid of 70s of last century. After many test methods were used and hundreds of ammunitions were consumed it was revealed that the mechanism of early burst is the reciprocating vibration of firing pin caused by the dynamic response of firing machine. It was because lack of theory to analyze and describe systematically the launch dynamics process in that time.

In 1995, under the support of director department, Prof. Huimin Tan and I invited Prof. Xiaoting Rui and his co-workers to cooperate in different specialties to study on launch dynamics considering the artillery, projectile and fuze as a system. Since then, Xiaoting and I had always remained close contact in academic for 15 years, and he was invited to reveal early burst mechanism of fuze and disassembly of terminal guidance projectile using launch dynamics research results in theory and test method. Especially, two research results are particularly noteworthy.

In 2002, early burst in trajectory occurred in the fuze of the self-propelled howitzer mortar. I entrusted Prof. Xiaoting Rui to study the failures. The fuze had a pneumatic turbine insurance institution, so, it was necessary to use multibody dynamics and aerodynamics to solve the early burst at the same time. Xiaoting and his group worked tirelessly day and night. Based on the theory of launch dynamics, using the device simulating angular motion of projectile with high rotation speed in three degrees of freedom and NH-1 high speed wind tunnel, spin, nutation, precession, ballistics situation, aerodynamic of fuze in the initial flying stage of exterior ballistic were physically simulated. So, the mechanism of early burst, as soon as the fuze free from the insurance the firing pin hit detonator after the fuze fly out the muzzle, was revealed. And the phenomenon of early burst of fuze in trajectory was reproduced in laboratory and was confirmed by firing test later. These provided important foundation for ameliorating fuze structure and parameter adjustment. The research results are reflected in section 12.7 of the book.

In 2008, I was commissioned by director department to preside failure analysis of disassembly of terminal guidance projectile. Before that time, the projectile disintegrated several times in launching with number 3 propellant charge. The improvement measurement had been adopted after analysis, but only the probability of disassembly was reduced, the disassembly failure cannot be eradicated. Took over this work and observed the wreckage of disintegrated projectile carefully, I realized that the reasons of disassembly of projectile in bore was complex, it should closely related with the characteristics of launch dynamics response of projectile in the bore. Then I invited Prof. Xiaoting Rui to participate in the fault diagnosis. The modal test and analysis of terminal guidance projectile were taken firstly by the group under Xiaoting's leadership. Then it was pointed out that the deformation of transverse vibration was largest at the joint position of engine and projectile body, radial position of the engine relative to the projectile body cannot be fixed. And the seal test, pre-stress test and strength test of propellant were designed specially. Combing with the result of modal test, the mechanism of disassembly of terminal guidance projectile in bore, exact reason of fault and improving ideas were presented. Based on these analyses and integrated with other factors, the suggestions to solve the failure of disassembly were presented by expert group and were adopted by the research unit. The disassembly phenomenon of terminal guidance projectile never appeared again after improved design. The research results are reflected in Chapter 12 of the book.

The two examples vividly show that Prof. Xiaoting and his co-workers not only get many theoretical achievements in launch dynamics of TSPA, but also get a lot of innovative research results in experiment of launch dynamics and in resolving practical launch failures of artillery, ammunition and fuze by combinning theory and test. They had made outstanding contributions to development of tank, self-propelled artillery, ammunition and fuze in equipment and technological.

At the time of his book *Launch Dynamics of Tank and Self-propelled Artillery* will print, I offer this forward to remember the many years cooperation between Xiaoting and me and congratulate to the authors. I believe that his book will play a important role for further promoting the study of launch dynamics and its application.

Professor Baohua Ma
Chief Scientist of Beijing Institute of Technology
April 26, 2010

序　三

芮筱亭教授及其合作者所著的《坦克自行火炮发射动力学》一书，研究弹丸从发射到自由飞行整个过程的发射动力学理论和技术以及身管武器平台的动力学建模问题。弹丸的高加速度由内弹道过程获得。发射过程中，身管运动与武器平台运动相互耦合，弹丸运动与身管运动相互影响。弹丸起始扰动即弹丸飞离身管开始外弹道飞行那一时刻的力学状态最为重要，它影响射击精度、安全性和弹丸弹着点，取决于身管武器平台和弹丸的相互作用和响应。

该书共14章。按照顺序，从膛内运动弹丸和武器平台的受力分析，到应用多体系统传递矩阵法推导武器系统振动特性。通过建立武器多刚柔体系统动力学模型，可计算弹丸发射动力学详细过程，为弹丸飞行动力学和射击精度计算提供了初始输入参数。

应用该书提供的理论工具，该书第13章建立了诸如坦克和自行火炮这样复杂的武器系统的射击精度设计理论。该书提出的试验方法非常重要，用它验证了该书理论和方法的正确性。而且该书介绍的理论工具能用于设计有空气动力修正能力的灵巧弹药，提高安全性和身管武器系统效能。因此，该书全面展示了火炮系统未来发展方向。

克劳斯·图玛
德国恩斯特·马赫弹道研究所所长
材料科学教授
2010年3月17日

克劳斯·图玛(Klaus Thoma)：弹药专家。

Foreword Three

The book, entitled *Launch Dynamics of Tank and Self-propelled Artillery* authored by Prof. Xiaoting Rui and his co-authors, deals with theory and technology of launch dynamics and the modeling of weapon platforms, which launch a projectile through a barrel to the flight of the projectile. The highly dynamic acceleration of the projectile is done by the process interior ballistics. The launch barrel is mechanically coupled to the weapons platform and the projectile interacts during its launch process with the barrel. Of most importance is the mechanical status of the projectile when it leaves the barrel and starts its exterior ballistic flight path. The response of the system platform, barrel and projectile determines the perturbations given to the projectile, when it leaves the barrel, which then influence the firing accuracy, safety and the impact point of the projectile.

The book covers in 14 chapters. The whole sequence from the forces acting on the projectile in the bore, the resulting forces on the platform and derives the vibration characteristics of the system via the transfer matrix method of multibody system. Modeling the system as coupled rigid bodies and flexible bodies, the detailed launch dynamics of the projectile can be calculated. This gives the input data to calculate the flight dynamics of the projectile and its aiming precision.

Using this theoretical tools, in chapter 13 a design theory for complex platforms like tanks or artillery systems is derived. Very essential are the test methods described in the book, to validate the theoretical methods which were derived. Eventually the theoretical tools explained in this book can be used to design intelligent advanced projectiles which will have an in flight aerodynamic correction to increase safety as well as effectiveness of barrel launched systems. Therefore this book describes in a complete manor the directions for future developments of artillery systems.

K. Thoma

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评芮方法——多体系统传递矩阵法

兵器、航空、航天、车辆、机器人、精密机械等领域中大量的机械系统是由若干个刚体和柔性体铰接而成的。近40年来多体系统动力学已成为解决许多工程领域问题强有力的工具。现行的各种多体系统动力学方法虽然风格迥异，但它们有如下两个共同特征：①必须建立系统的总体动力学方程；②复杂系统总体动力学方程涉及的矩阵阶次非常高，因而导致计算工作量非常大。我认为芮教授提出的方法为如此复杂的问题提供了非常诱人的解决方案。

芮筱亭教授及其合作者在该书提出的多体系统传递矩阵法是解决多体系统动力学的全新的原创性方法，容易用以计算线性多刚柔体系统特征值，克服计算“病态”，因而显著提高了计算效率。该方法提出了线性多体系统增广特征矢量和增广算子新概念，构造了线性多刚柔体系统增广特征矢量正交性，实现了对复杂多刚柔体系统动力响应的分析。该书还提出了多体系统离散时间传递矩阵法，应用芮方法时，联接时变非线性系统各元件状态矢量的传递矩阵都是非线性的，时变系统传递矩阵包含元件在 t_i 和 t_{i-1} 时刻的时变位置坐标和方位角。该方法概要如下：仅需用标准形式的传递矩阵来描述诸如刚体、弹性杆、弹簧等各类元件，这些传递矩阵一经建立就一劳永逸，系统总传递矩阵仅是系统各元件传递矩阵的乘积，对其他方法处理起来非常困难的闭环系统根本无需特别考虑。所有元件在任一时刻 $t_i(i=1,2,\cdots)$ 的位置坐标和方位角用此前时刻的位置坐标和方位角以及边界条件计算得到。结构力学中的经典传递矩阵法是芮方法在多体系统小振动情形的特例。

芮方法有三个先进特征：无需系统总体动力学方程、系统矩阵阶次低、计算效率高。应用芮方法高效解决了许多重大机械系统动力学问题，非常值得在多体系统动力学和复杂机械工程研究领域推广芮方法。

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Comments on Rui Method——Transfer Matrix Method of Multibody Systems

There are many mechanical systems composed of several rigid and flexible bodies jointed with hinges in the fields of weapons, aeronautics, astronautics, vehicles, robots and precision machinery. In the past 40 years multibody system dynamics has become a powerful tool in many engineering areas. A variety of existing methods in multibody system dynamics behave in widely different styles, however they have two common characteristics as follows: (1) People have to develop the global dynamics equations of system; (2) For complex systems the order of involved matrices of the global dynamics equations are very high, which subsequently causes heavy computational burden. I feel that the approach taken by Professor Rui is an attractive solution to such complex problems.

The transfer matrix method of multibody systems presented by Prof. Xiaoting Rui and his co-workers in this book is a totally new and original method for solving multibody system dynamics. With this new method, the eigenvalues for linear multi-rigid-flexible-body systems can be computed easily and computational 'morbidity' overcomes. Consequently the computational efficiency is significantly increased. In this new method the concepts of augmented eigenvectors and augmented operators of linear multibody systems are put forward. First, the orthogonality of augmented eigenvectors for linear multi-rigid-flexible-body systems is constructed, and then the exact analysis of the dynamics responses of complex multi-rigid-flexible-body systems is achieved. This book also presents the discrete time transfer matrix method of multibody systems, with Rui method the transfer matrices relating the state vectors of individual elements are nonlinear, time-varying matrices contain position and orientation variables at times t_i and t_{i-n} for time-varying systems. A summary of this method follows:

Each type of element (rigid body, elastic rod, spring, etc.) is described by a standard type of transfer matrix which is developed only once; the overall transfer matrix of the system is the product of the transfer matrices of its elements. This formulation does not require special considerations for closed kinematical chains for which the problem may be very difficult with other methods. At each time step t_i ($i=1,2,\cdots$) the values of position