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Friedrich Sauvigny

Partial Differential Equations 2

Functional Analytic Methods

偏微分方程 第2卷

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With Consideration of Lectures
by E. Heinz

Springer

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Introduction to Volume 2 – Functional Analytic Methods

In this second volume, FUNCTIONAL ANALYTIC METHODS, we continue our textbook PARTIAL DIFFERENTIAL EQUATIONS OF GEOMETRY AND PHYSICS. From both areas we shall answer central questions such as curvature estimates or eigenvalue problems, for instance. With the title of our textbook we also want to emphasize the pure and applied aspects of partial differential equations. It turns out that the concepts of solutions are permanently extended in the theory of partial differential equations. Here the classical methods do not lose their significance. Besides the n-dimensional theory we equally want to present the two-dimensional theory – so important to our geometric intuition.

We shall solve the differential equations by the continuity method, the variational method or the topological method. The continuity method may be preferred from a geometric point of view, since the stability of the solution is investigated there. The variational method is very attractive from the physical point of view; however, difficult regularity questions for the weak solution appear with this method. The topological method controls the whole set of solutions during the deformation of the problem, and does not depend on uniqueness as does the variational method.

We would like to mention that this textbook is a translated and expanded version of the monograph by *Friedrich Sauvigny: Partielle Differentialgleichungen der Geometrie und der Physik 2 – Funktionalanalytische Lösungsmethoden – Unter Berücksichtigung der Vorlesungen von E.Heinz*, which appeared in *Springer-Verlag* in 2005.

In Chapter VII we consider – in general – nonlinear operators in Banach spaces. With the aid of Brouwer's degree of mapping from Chapter III we prove Schauder's fixed point theorem in § 1 ; and we supplement Banach's fixed point theorem. In § 2 we define the Leray-Schauder degree for mappings in Banach spaces by a suitable approximation, and we prove its fundamental properties in § 3 . In this section we refer to the lecture [H4] of my academic teacher, Professor Dr. E. Heinz in Göttingen.

Then, by transition to linear operators in Banach spaces, we prove the fundamental solution-theorem of F. Riesz via the Leray-Schauder degree. At the end of this chapter we derive the Hahn-Banach continuation theorem by Zorn's lemma(compare [HS]).

In Chapter VIII on Linear Operators in Hilbert Spaces, we transform the eigenvalue problems of Sturm-Liouville and of H. Weyl for differential operators into integral equations in § 1 . Then we consider weakly singular integral operators in § 2 and prove a theorem of I. Schur on iterated kernels. In § 3 we further develop the results from Chapter II, § 6 on the Hilbert space and present the abstract completion of pre-Hilbert-spaces. Bounded linear operators in Hilbert spaces are treated in § 4: The continuation theorem, Adjoint and Hermitian operators, Hilbert-Schmidt operators, Inverse operators, Bilinear forms and the theorem of Lax-Milgram are presented. In § 5 we study the transformation of Fourier-Plancherel as a unitary operator on the Hilbert space $L^2(\mathbb{R}^n)$.
Completely continuous, respectively compact operators are studied in § 6 together with weak convergence. The operators with finite square norms represent an important example. The solution-theorem of Fredholm on operator equations in Hilbert spaces is deduced from the corresponding result of F. Riesz in Banach spaces. We particularly apply these results to weakly singular integral operators.
In § 7 we prove the spectral theorem of F. Rellich on completely continuous and Hermitian operators by variational methods. Then we address the Sturm-Liouville eigenvalue problem in § 8 and expand the relevant integral kernels into their eigenfunctions. Following ideas of H. Weyl we treat the eigenvalue problem for the Laplacian on domains in $\mathbb{R}^n$ by the integral equation method in § 9. In this chapter as well, we take a lecture of Professor Dr. E. Heinz into consideration (compare [H3]). For the study of eigenvalue problems we recommend the classical treatise [CH] of R. Courant and D. Hilbert, which has also smoothed the way into modern physics.

We have been guided into functional analysis with the aid of problems concerning differential operators in mathematical physics (compare [He1] and [He2]). The usual content of functional analysis can be taken from the Chapters II §§ 6-8, VII and VIII. Additionally, we investigated the solvability of nonlinear operator equations in Banach spaces. For the spectral theorem of unbounded, selfadjoint operators we refer the reader to the literature.

In our compendium we shall directly construct classical solutions of boundary and initial value problems for linear and nonlinear partial differential equations with the aid of functional analytic methods. By appropriate a priori estimates with respect to the Hölder norm we establish the existence of solutions in classical function spaces.

In Chapter IX, §§ 1-3 , we essentially follow the book of I. N. Vekua [V] and solve the Riemann-Hilbert boundary value problem by the integral equation method. Using the lecture [H6] , we present Schauder's continuity method in §§ 4-7 in order to solve boundary value problems for linear elliptic differential equations with n independent variables. Therefore, we completely prove the Schauder estimates.

In Chapter X on weak solutions of elliptic differential equations, we profit from the *Grundlehren* [GT] Chapters 7 and 8 of D. Gilbarg and N. S. Trudinger. Here, we additionally recommend the textbook [Jo] of J. Jost and the compendium [E] by L. C. Evans.
We introduce Sobolev spaces in § 1 and prove the embedding theorems in § 2. Having established the existence of weak solutions in § 3 , we show the boundedness of weak solutions by Moser's iteration method in § 4 . Then we investigate Hölder continuity of weak solutions in the interior and at the boundary; see §§ 5-7 . Restricting ourselves to interesting classes of equations, we can illustrate the methods of proof in a transparent way. Finally, we apply the results to equations in divergence form; see § 8, § 9, and § 10.

In Chapter XI, §§ 1-2, we concisely lay the foundations of differential geometry (compare [BL]) and of the calculus of variations. Then, we discuss the theory of characteristics for nonlinear hyperbolic differential equations in two variables (compare [CH], [G], [H5]) in § 3 and § 4. In particular, we solve the Cauchy initial value problem via Banach's fixed point theorem. In § 6 we present H. Lewy's ingenious proof for the analyticity theorem of S. Bernstein. Here, we would like to refer the reader to the textbook by P. Garabedian [G] as well.

On the basis of Chapter IV from Volume 1, Generalized Analytic Functions, we treat Nonlinear Elliptic Systems in Chapter XII. We give a detailed survey of the results at the beginning of this chapter.
Having presented Jäger's maximum principle in § 1 , we develop the general theory in §§ 2-5 from the fundamental treatise of E. Heinz [H7] about nonlinear elliptic systems. An existence theorem for nonlinear elliptic systems is situated in the center, which is gained by the Leray-Schauder degree. In §§ 6-10 we apply the results to differential geometric problems. Here, we introduce conformal parameters into a nonanalytic Riemannian metric by a nonlinear continuity method. We directly establish the necessary a priori estimates which extend to the boundary. Finally, we solve the Dirichlet problem for nonparametric equations of prescibed mean curvature by the uniformization method. For this chapter, one should also study the *Grundlehren* [DHKW], especially Chapter 7, by U. Dierkes and S. Hildebrandt, where the theory of minimal surfaces is presented. With the aid of nonlinear elliptic systems we can also study the Monge-Ampère differential equation, which is not quasilinear any more. This theory has been developed by H. Lewy, E. Heinz and F. Schulz (vgl. [Sc]) in order to solve Weyl's embedding problem.

This textbook PARTIAL DIFFERENTIAL EQUATIONS has been developed from lectures, which I have been giving in the Brandenburgische Technische Universität at Cottbus since the winter semester 1992/93. The monograph , in part, builds upon the lectures of Professor Dr. E. Heinz, whom I was fortunate to know as his student in Göttingen from 1971 to 1978. As an assistant in Aachen from 1978 to 1983, I very much appreciated the elegant lecture cycles of Professor Dr. G. Hellwig. Since my research visit to Bonn in 1989/90, Professor Dr. S. Hildebrandt has followed my academic activities with his supportive interest. All of them will forever have my sincere gratitude!

My thanks go also to M. Sc. Matthias Bergner for his elaboration of Chapter IX. Dr. Frank Müller has excellently worked out the further chapters, and he has composed the whole TEX-manuscript. I am cordially grateful for his great scientific help. Furthermore, I owe to Mrs. Prescott valuable suggestions to improve the style of the language. Moreover, I would like to express my gratitude to the referee of the English edition for his proposal, to add some historical notices and pictures, as well as to Professor Dr. M. Fröhner for his help, to incorporate the graphics into this textbook. Finally, I thank Herrn C. Heine and all the other members of Springer-Verlag for their collaboration and confidence.

Last but not least, I would like to acknowledge gratefully the continuous support of my wife, Magdalene Frewer-Sauvigny in our University Library and at home.

Cottbus, in May 2006 *Friedrich Sauvigny*

Contents of Volume 1 – Foundations and Integral Representations

Contents of Volume 2 – Functional Analytic Methods

VII

Operators in Banach Spaces

We shall now present methods from the nonlinear functional analysis. In this chapter we build upon our deliberations from Chapter II, §§ 6-8. A detailed account of the contents for this chapter is given in the 'Introduction to Volume 2' above.

§1 Fixed point theorems

Definition 1. *The* Banach space $\mathcal{B}$ *is a linear normed complete (infinite-dimensional) vector space above the field of real numbers* $\mathbb{R}$.

Example 1. Let the set $\Omega \subset \mathbb{R}^n$ be open, $1 \leq p < +\infty$, $\mathcal{B} := L^p(\Omega)$. We have $f \in L^p(\Omega)$ if and only if $f : \Omega \to \mathbb{R}$ is measurable and

$$\int_\Omega |f(x)|^p \, dx < +\infty$$

holds true. For the element $f \in \mathcal{B}$ we define the norm

$$\|f\| := \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

We obtain the *Lebesgue space* with $\mathcal{B}$. The case $p = 2$ reduces to the Hilbert space using the inner product

$$(f, g) := \int_\Omega f(x) g(x) \, dx.$$

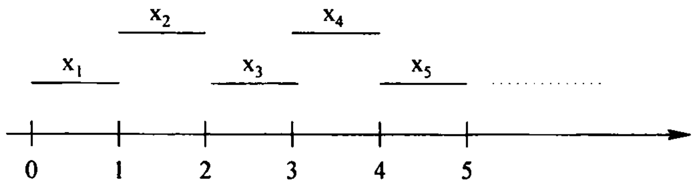
Example 2. (Hilbert's sequence space ℓ^p) For the sequence $x = (x_1, x_2, x_3, \ldots)$ we have $x \in \ell^p$ with $1 \leq p < +\infty$ if and only if

$$\sum_{i=1}^{\infty} |x_i|^p < +\infty$$

is fulfilled. By the norm

$$\|x\| := \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}$$

the set ℓ^p becomes a Banach space. Obviously, we have $\ell^p \subset L^p((0,+\infty))$.



Example 3. (Sobolev spaces) Let the numbers $k \in \mathbb{N}$, $1 \leq p < +\infty$ be given, and $\Omega \subset \mathbb{R}^n$ denotes an open set. The space

$$\mathcal{B} = W^{k,p}(\Omega) := \Big\{ f : \Omega \to \mathbb{R} \; : \; D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq k \Big\}$$

with the norm

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha f(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad f \in \mathcal{B},$$

represents a Banach space. Here, the vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ indicates a multi-index, and we set

$$|\alpha| := \sum_{i=1}^{n} \alpha_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

In this context we refer the reader to Chapter X, § 1.

Example 4. Finally, we consider the *classical Banach spaces* $C^k(\overline{\Omega})$, $k = 0, 1, 2, 3, \ldots$, on a bounded domain $\Omega \subset \mathbb{R}^n$. We have $f \in C^k(\overline{\Omega})$ if and only if

$$\sup_{x \in \Omega} \sum_{|\alpha| \leq n} |D^\alpha f(x)| < +\infty$$

holds true. Here $\alpha \in \mathbb{N}_0^n$ again denotes a multi-index. The vector space $\mathcal{B} := C^k(\overline{\Omega})$ equipped with the norm

$$\|f\|_{C^k(\overline{\Omega})} := \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha f(x)|$$

is complete, and consequently represents a Banach space. Here, we abbreviate

$$D^\alpha f(x) := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} f(x), \qquad \alpha \in \mathbb{N}_0^n, \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Definition 2. *A subset $K \subset \mathcal{B}$ of the Banach space $\mathcal{B}$ is named* convex, *if we have the inclusion $\lambda x + (1-\lambda)y \in K$ for each two points $x, y \in K$ and each parameter $\lambda \in [0,1]$.*

Remarks:

1. When K is closed, this set is convex if and only if
$$x, y \in K \quad \Rightarrow \quad \frac{1}{2}(x+y) \in K$$
holds true.
2. For a convex set K we have the following implication: Choosing the points $x_1, \ldots, x_n \in K$ and the parameters $\lambda_i \geq 0$, $i = 1, \ldots, n$ with $\lambda_1 + \ldots + \lambda_n = 1$, we infer
$$\sum_{i=1}^{n} \lambda_i x_i \in K.$$

Definition 3. *A subset $E \subset \mathcal{B}$ is called* precompact, *if each sequence*
$$\{x_n\}_{n=1,2,\ldots} \subset E$$
contains a Cauchy sequence as a subsequence. If the set E is additionally closed, which means $\{x_n\}_{n \in \mathbb{N}} \subset E$ with $x_n \to x$ for $n \to \infty$ in $\mathcal{B}$ implies $x \in E$, we call the set E compact.

Example 5. Let $E \subset \mathcal{B}$ be a closed and bounded subset of a finite-dimensional subspace of $\mathcal{B}$. Then the Weierstraß selection theorem yields that E is compact.

Example 6. For infinite-dimensional Banach spaces, bounded and closed subsets are not necessarily compact: Choosing $k \in \mathbb{N}$ we consider the set of sequences $x_k := (\delta_{kj})_{j=1,2,\ldots}$ in the space ℓ^2. As usual, δ_{kj} denotes the Kronecker symbol. Obviously, we have $\|x_k\| = 1$ for $k \in \mathbb{N}$ and
$$\|x_k - x_l\| = \sqrt{2}\,(1 - \delta_{kl}) \qquad \text{for all} \quad k, l \in \mathbb{N}.$$
Therefore, the set $\{x_k\}_{k=1,2,\ldots}$ is not precompact.

Example 7. A bounded set in $C^k(\overline{\Omega})$ is compact, if we additionally require a modulus of continuity for the k-th partial derivatives: Consider the set
$$E := \left\{ f \in C^k(\overline{\Omega}) \,:\, \begin{array}{c} \|f\|_{C^k(\overline{\Omega})} \leq M; \\ |D^\alpha f(x) - D^\alpha f(y)| \leq M'|x-y|^\vartheta \\ \text{for all} \quad x, y \in \overline{\Omega}, \ |\alpha| = k \end{array} \right\}$$
with $k \in \mathbb{N}_0$, $M, M' \in (0, +\infty)$ and $\vartheta \in (0,1]$. By the Theorem of Arzelà-Ascoli we easily deduce that the set
$$E \subset \mathcal{B} := C^k(\overline{\Omega})$$
is compact.

Definition 4. *On the subset $E \subset \mathcal{B}$ in the Banach space $\mathcal{B}$ we have defined the mapping $F : E \to \mathcal{B}$. We call F* continuous, *if*

$$x_n \to x \quad for \quad n \to \infty \quad in \quad E$$

implies

$$F(x_n) \to F(x) \qquad for \quad n \to \infty \quad in \quad \mathcal{B}.$$

We name F completely continuous *(or* compact *as well), if additionally the set $F(E) \subset \mathcal{B}$ is precompact; this means all sequences $\{x_n\}_{n=1,2,\ldots} \subset E$ contain a subsequence $\{x_{n_k}\}_k \subset \{x_n\}_n$, such that $\{F(x_{n_k})\}_{k=1,2,\ldots}$ gives a Cauchy sequence in $\mathcal{B}$.*

Proposition 1. *Let K be a precompact subset of the Banach space $\mathcal{B}$. For all $\varepsilon > 0$ we have finitely many elements $w_1, \ldots, w_N \in K$ with $N = N(\varepsilon) \in \mathbb{N}$, such that the covering property*

$$K \subset \bigcup_{j=1}^{N(\varepsilon)} \left\{ x \in \mathcal{B} : \|x - w_j\| \leq \frac{\varepsilon}{2} \right\}$$

is fulfilled.

Proof: We choose $w_1 \in K$ and the covering property is already valid if

$$K \subset \left\{ x \in \mathcal{B} : \|x - w_1\| \leq \frac{\varepsilon}{2} \right\}$$

holds true. When this is not the case, there exists a further point $w_2 \in K$ with $\|w_2 - w_1\| > \frac{\varepsilon}{2}$ and we consider the balls

$$\left\{ x \in \mathcal{B} : \|x - w_j\| \leq \frac{\varepsilon}{2} \right\} \qquad \text{for} \quad j = 1, 2.$$

If they do not yet cover the set K , there would exist a third point $w_3 \in K$ with $\|w_3 - w_j\| > \frac{\varepsilon}{2}$ for $j = 1, 2$. In case the procedure did not stop, we could find a sequence $\{w_j\}_{j=1,2\ldots} \subset K$ of points satisfying

$$\|w_j - w_i\| > \frac{\varepsilon}{2} \qquad \text{for} \quad i = 1, \ldots, j-1.$$

This yields a contradiction to the precompactness of the set K. q.e.d.

Proposition 2. *Let K be a precompact set in $\mathcal{B}$, and $\varepsilon > 0$ is arbitrarily given. Then we have finitely many elements $w_1, \ldots, w_N \in K$ with $N = N(\varepsilon) \in \mathbb{N}$ continuous functions*

$$t_i = t_i(x) : \overline{K} \to \mathbb{R} \in C^0(\overline{K})$$

satisfying

$$t_i(x) \geq 0 \quad and \quad \sum_{i=1}^{N} t_i(x) = 1 \qquad in \quad K,$$

such that the following inequality holds true:

$$\left\| \sum_{i=1}^{N} t_i(x) w_i - x \right\| \leq \varepsilon \qquad for\ all \quad x \in \overline{K}.$$

Proof: We choose the points $\{w_1, \ldots, w_N\} \subset K$ according to Proposition 1. We define the continuous function $\varphi(\tau) : [0, +\infty) \to [0, +\infty)$ via

$$\varphi(\tau) := \begin{cases} \varepsilon - \tau, & \text{for } 0 \leq \tau \leq \varepsilon \\ 0, & \text{for } \varepsilon \leq \tau < +\infty \end{cases},$$

and obtain

$$\sum_{j=1}^{N} \varphi(\|x - w_j\|) \geq \frac{\varepsilon}{2} \qquad \text{for all} \quad x \in \overline{K}.$$

Consequently, the functions

$$t_i(x) := \frac{\varphi(\|x - w_i\|)}{\sum\limits_{j=1}^{N} \varphi(\|x - w_j\|)}, \qquad x \in \overline{K}, \quad i = 1, \ldots, N$$

are well-defined, and we note that

$$t_i \in C^0(\overline{K}, [0,1]) \qquad \text{and} \qquad \sum_{i=1}^{N} t_i(x) = 1 \quad \text{for all} \quad x \in \overline{K}.$$

Now, we can estimate as follows:

$$\begin{aligned} \left\| x - \sum_{i=1}^{N} t_i(x) w_i \right\| &= \left\| \sum_{i=1}^{N} t_i(x)(x - w_i) \right\| \\ &\leq \sum_{i=1}^{N} t_i(x) \|x - w_i\| \\ &\leq \sum_{i=1}^{N} t_i(x) \varepsilon = \varepsilon \qquad \text{for all} \quad x \in \overline{K}. \end{aligned}$$

This gives us the inequality stated. q.e.d.

Proposition 3. *Let the set $E \subset \mathcal{B}$ be closed and the function $F : E \to \mathcal{B}$ be completely continuous. To each number $\varepsilon > 0$ then we have $N = N(\varepsilon) \in \mathbb{N}$ elements $w_1, \ldots, w_N \in F(E)$ and N continuous functions $F_j : E \to \mathbb{R}$, $j = 1, \ldots, N$ satisfying*