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Friedrich Sauvigny

Partial Differential Equations 1

Foundations and Integral Representations

偏微分方程 第1卷

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Foundations and Integral Representations

With Consideration of Lectures
by E. Heinz

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Introduction to Volume 1 – Foundations and Integral Representations

Partial differential equations equally appear in physics and geometry. Within mathematics they unite the areas of complex analysis, differential geometry and calculus of variations. The investigation of partial differential equations has substantially contributed to the development of functional analysis. Though a relatively uniform treatment of ordinary differential equations is possible, quite multiple and diverse methods are available for partial differential equations. With this two-volume textbook we intend to present the entire domain PARTIAL DIFFERENTIAL EQUATIONS – so rich in theories and applications – to students at the intermediate level. We presuppose a basic knowledge of Analysis, as it is conveyed in S. Hildebrandt's very beautiful lectures [Hi1,2] or in the lecture notes [S1,2] or in W. Rudin's influential textbook [R]. For the convenience of the reader we develop further foundations from Analysis in a form adequate to the theory of partial differential equations. Therefore, this textbook can be used for a course extending over several semesters. A survey of all the topics treated is provided by the table of contents. For advanced readers, each chapter may be studied independently from the others.

Selecting the topics of our lectures and consequently for our textbooks, I tried to follow the advice of one of the first great scientists – of the Enlightenment – at the University of Göttingen, namely G.C. Lichtenberg: *Teach the students h o w they think and not w h a t they think!* As a student at this University, I admired the commemorative plates throughout the city in honor of many great physicists and mathematicians. In this spirit I attribute the results and theorems in our compendium to the persons creating them – to the best of my knowledge.

We would like to mention that this textbook is a translated and expanded version of the monograph by *Friedrich Sauvigny: Partielle Differentialgleichungen der Geometrie und der Physik 1 – Grundlagen und Integraldarstellungen – Unter Berücksichtigung der Vorlesungen von E. Heinz*, which appeared in *Springer-Verlag* in 2004.

In Chapter I we treat Differentiation and Integration on Manifolds, where we use the improper Riemannian integral. After the Weierstrassian approximation theorem in § 1 , we introduce differential forms in § 2 as functionals on surfaces – parallel to [R]. Their calculus rules are immediately derived from the determinant laws and the transformation formula for multiple integrals. With the partition of unity and an adequate approximation we prove the Stokes integral theorem for manifolds in § 4 , which may possess singular boundaries of capacity zero besides their regular boundaries. In § 5 we especially obtain the Gaussian integral theorem for singular domains as in [H1], which is indispensable for the theory of partial differential equations. After the discussion of contour integrals in § 6 , we shall follow [GL] in § 7 and represent A. Weil's proof of the Poincaré lemma. In § 8 we shall explicitly construct the $*$-operator for certain differential forms in order to define the Beltrami operators. Finally, we represent the Laplace operator in n-dimensional spherical coordinates.

In Chapter II we shall constructively supply the Foundations of Functional Analysis. Having presented Daniell's integral in § 1 , we shall continue the Riemannian integral to the Lebesgue integral in § 2. The latter is distinguished by convergence theorems for pointwise convergent sequences of functions. We deduce the theories of Lebesgue measurable sets and functions in a natural way; see § 3 and § 4. In § 5 we compare Lebesgue's with Riemann's integral. Then we consider Banach and Hilbert spaces in § 6 , and in § 7 we present the Lebesgue spaces $L^p(X)$ as classical Banach spaces. Especially important are the selection theorems with respect to almost everywhere convergence due to H. Lebesgue and with respect to weak convergence due to D. Hilbert. Following ideas of J. v. Neumann we investigate bounded linear functionals on $L^p(X)$ in § 8 . For this Chapter I have profited from a seminar on functional analysis, offered to us as students by my academic teacher, Professor Dr. E. Heinz in Göttingen.

In Chapter III we shall study topological properties of mappings in $\mathbb{R}^n$ and solve nonlinear systems of equations. In this context we utilize Brouwer's degree of mapping, for which E. Heinz has given an ingenious integral representation (compare [H8]). Besides the fundamental properties of the degree of mapping, we obtain the classical results of topology. For instance, the theorems of Poincaré on spherical vector-fields and of Jordan-Brouwer on topological spheres in $\mathbb{R}^n$ appear. The case $n = 2$ reduces to the theory of the winding number. In this chapter we essentially follow the first part of the lecture on fixed point theorems [H4] by E. Heinz.

In Chapter IV we develop the theory of holomorphic functions in one and several complex variables. Since we utilize the Stokes integral theorem, we easily attain the well-known theorems from the classical theory of functions in § 2 and § 3. In the subsequent paragraphs we additionally study solutions of the inhomogeneous Cauchy-Riemann differential equation, which has been completely investigated by L. Bers and I. N. Vekua (see [V]) . In § 6 we assemble statements on pseudoholomorphic functions, which are similar to holomorphic functions as far as the behavior at their zeroes is concerned. In § 7 we prove the Riemannian mapping theorem with an extremal method due to Koebe and investigate in § 8 the boundary behavior of conformal mappings. In this chapter we intend to convey, to some degree, the splendor of the lecture [Gr] by H. Grauert on complex analysis.

Chapter V is devoted to the study of Potential Theory in $\mathbb{R}^n$. With the aid of the Gaussian integral theorem we investigate Poisson's differential equation in § 1 and § 2 , and we establish an analyticity theorem. With Perron's method we solve the Dirichlet problem for Laplace's equation in § 3. Starting with Poisson's integral representation we develop the theory of spherical harmonic functions in $\mathbb{R}^n$; see § 4 and § 5 . This theory was founded by Legendre, and we owe this elegant representation to G. Herglotz. In this chapter as well, I was able to profit decisively from the lecture [H2] on partial differential equations by my academic teacher, Professor Dr. E. Heinz in Göttingen.

In Chapter VI we consider linear partial differential equations in $\mathbb{R}^n$. We prove the maximum principle for elliptic differential equations in § 1 and apply this central tool on quasilinear, elliptic differential equations in § 2 (compare the lecture [H6]). In § 3 we turn to the heat equation and present the parabolic maximum-minimum principle. Then in § 4 , we comprehend the significance of characteristic surfaces and establish an energy estimate for the wave equation. In § 5 we solve the Cauchy initial value problem of the wave equation in $\mathbb{R}^n$ for the dimensions $n = 1, 3, 2$. With the aid of Abel's integral equation we solve this problem for all $n \geq 2$ in § 6 (compare the lecture [H5]). Then we consider the inhomogeneous wave equation and an initial-boundary-value problem in § 7 . For parabolic and hyperbolic equations we recommend the textbooks [GuLe] and [J]. Finally, we classify the linear partial differential equations of second order in § 8. We discover the Lorentz transformations as invariant transformations for the wave equation (compare [G]).
With Chapters V and VI we intend to give a geometrically oriented introduction into the theory of partial differential equations without assuming prior functional analytic knowledge.

It is a pleasure to express my gratitude to Dr. Steffen Fröhlich and to Dr. Frank Müller for their immense help with taking the lecture notes in the Brandenburgische Technische Universität Cottbus, which are basic to this monograph. For many valuable hints and comments and the production of the whole TEX-manuscript I express my cordial thanks to Dr. Frank Müller. He has elaborated this textbook in a superb way.

Furthermore, I owe to Mrs. Prescott valuable recommendations to improve the style of the language. Moreover, I would like to express my gratitude to the referee of the English edition for his proposal, to add some historical notices and pictures, as well as to Professor Dr. M. Fröhner for his help, to incorporate the graphics into this textbook. Finally, I thank Herrn C. Heine and all the other members of Springer-Verlag for their collaboration and confidence.

Last but not least, I would like to acknowledge gratefully the continuous support of my wife, Magdalene Frewer-Sauvigny in our University Library and at home.

Cottbus, in May 2006 *Friedrich Sauvigny*

Contents of Volume 2 – Functional Analytic Methods

Contents of Volume 1 – Foundations and Integral Repesentations

I

Differentiation and Integration on Manifolds

In this chapter we lay the foundations for our treatise on partial differential equations. A detailed description for the contents of Chapter I is given in the Introduction to Volume 1 above. At first, we fix some familiar notations used throughout the two volumes of our textbook.

By the symbol $\mathbb{R}^n$ we denote the n-dimensional Euclidean space with the points $x = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{R}$, and we define their modulus

$$|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

In general, we denote open subsets in $\mathbb{R}^n$ by the symbol Ω. By the symbol $\overline{M}$ we indicate the topological closure and by $\overset{\circ}{M}$ the open kernel of a set $M \subset \mathbb{R}^n$. In the sequel, we shall use the following linear spaces of functions:

- $C^0(\Omega)$ ...... continuous functions on Ω
- $C^k(\Omega)$ ...... k-times continuously differentiable functions on Ω
- $C_0^k(\Omega)$ ...... k-times continuously differentiable functions f on Ω with the compact support $\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}} \subset \Omega$
- $C^k(\overline{\Omega})$ ...... k-times continuously differentiable functions on Ω, whose derivatives up to the order k can be continuously extended onto the closure $\overline{\Omega}$
- $C_0^k(\Omega \cup \Theta)$.. k-times continuously differentiable functions f on Ω, whose derivatives up to the order k can be extended onto the closure $\overline{\Omega}$ continuously with the property $\operatorname{supp} f \subset \Omega \cup \Theta$
- $C_*^*(*, K)$ ... space of functions as above with values in $K = \mathbb{R}^n$ or $K = \mathbb{C}$.

Finally, we utilize the notations

- ∇u .... gradient $(u_{x_1}, \ldots, u_{x_n})$ of a function $u = u(x_1, \ldots, x_n) \in C^1(\mathbb{R}^n)$

Δu .... Laplace operator $\sum_{i=1}^{n} u_{x_i x_i}$ of a function $u \in C^2(\mathbb{R}^n)$

J_f .... functional determinant or *Jacobian* of a function $f : \mathbb{R}^n \to \mathbb{R}^n \in C^1(\mathbb{R}^n, \mathbb{R}^n)$.

§1 The Weierstraß approximation theorem

Let $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ denote an open set and $f(x) \in C^k(\Omega)$ with $k \in \mathbb{N} \cup \{0\} =: \mathbb{N}_0$ a k-times continuously differentiable function. We intend to prove the following statement:
There exists a sequence of polynomials $p_m(x)$, $x \in \mathbb{R}^n$ for $m = 1, 2, \ldots$ which converges on each compact subset $C \subset \Omega$ uniformly towards the function $f(x)$. Furthermore, all partial derivatives up to the order k of the polynomials p_m converge uniformly on C towards the corresponding derivatives of the function f. The coefficients of the polynomials p_m depend on the approximation, in general. If this were not the case, the function

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x^2}\right), & x > 0 \\ 0, & x \le 0 \end{cases}$$

could be expanded into a power series. However, this leads to the evident contradiction:

$$0 \equiv \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad .$$

In the following Proposition, we introduce a 'mollifier' which enables us to smooth functions.

Proposition 1. *We consider the following function to each $\varepsilon > 0$, namely*

$$\begin{aligned} K_\varepsilon(z) &:= \frac{1}{\sqrt{\pi\varepsilon}^{\,n}} \exp\left(-\frac{|z|^2}{\varepsilon}\right) \\ &= \frac{1}{\sqrt{\pi\varepsilon}^{\,n}} \exp\left(-\frac{1}{\varepsilon}(z_1^2 + \ldots + z_n^2)\right), \qquad z \in \mathbb{R}^n. \end{aligned}$$

Then this function $K_\varepsilon = K_\varepsilon(z)$ possesses the following properties:

1. *We have $K_\varepsilon(z) > 0$ for all $z \in \mathbb{R}^n$;*
2. *The condition $\int_{\mathbb{R}^n} K_\varepsilon(z)\, dz = 1$ holds true;*
3. *For each $\delta > 0$ we observe: $\lim_{\varepsilon \to 0+} \int_{|z| \ge \delta} K_\varepsilon(z)\, dz = 0$.*

Proof:

1. The exponential function is positive, and the statement is obvious.
2. We substitute $z = \sqrt{\varepsilon}x$ with $dz = \sqrt{\varepsilon}^n\, dx$ and calculate

$$\int_{\mathbb{R}^n} K_\varepsilon(z)\, dz = \frac{1}{\sqrt{\pi\varepsilon}^n} \int_{\mathbb{R}^n} \exp\left(-\frac{|z|^2}{\varepsilon}\right) dz$$

$$= \frac{1}{\sqrt{\pi}^n} \int_{\mathbb{R}^n} \exp\Big(-|x|^2\Big)\, dx = \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp\Big(-t^2\Big)\, dt\right)^n = 1.$$

3. We utilize the substitution from part 2 of our proof and obtain

$$\int_{|z|\geq\delta} K_\varepsilon(z)\, dz = \frac{1}{\sqrt{\pi}^n} \int_{|x|\geq\delta/\sqrt{\varepsilon}} \exp\Big(-|x|^2\Big)\, dx \longrightarrow 0 \qquad \text{for} \quad \varepsilon \to 0+.$$

q.e.d.

Proposition 2. *Let us consider $f(x) \in C_0^0(\mathbb{R}^n)$ and additionally the function*

$$f_\varepsilon(x) := \int_{\mathbb{R}^n} K_\varepsilon(y-x) f(y)\, dy, \qquad x \in \mathbb{R}^n$$

for $\varepsilon > 0$. Then we infer

$$\sup_{x\in\mathbb{R}^n} |f_\varepsilon(x) - f(x)| \longrightarrow 0 \quad \textit{for} \quad \varepsilon \to 0+,$$

and consequently the functions $f_\varepsilon(x)$ converge uniformly on the space $\mathbb{R}^n$ towards the function $f(x)$.

Proof: On account of its compact support, the function $f(x)$ is uniformly continuous on the space $\mathbb{R}^n$. The number $\eta > 0$ being given, we find a number $\delta = \delta(\eta) > 0$ such that

$$x, y \in \mathbb{R}^n, \; |x-y| \leq \delta \quad \Longrightarrow \quad |f(x) - f(y)| \leq \eta.$$

Since f is bounded, we find a quantity $\varepsilon_0 = \varepsilon_0(\eta) > 0$ satisfying

$$2 \sup_{y\in\mathbb{R}^n} |f(y)| \int_{|y-x|\geq\delta} K_\varepsilon(y-x)\, dy \leq \eta \quad \text{for all} \quad 0 < \varepsilon < \varepsilon_0.$$

We note that

$$|f_\varepsilon(x) - f(x)| = \left| \int_{\mathbb{R}^n} K_\varepsilon(y-x)\, f(y)\, dy - f(x) \int_{\mathbb{R}^n} K_\varepsilon(y-x)\, dy \right|$$

$$\leq \left| \int_{|y-x|\leq\delta} K_\varepsilon(y-x)\, \{f(y) - f(x)\}\, dy \right|$$

$$+ \left| \int_{|y-x|\geq\delta} K_\varepsilon(y-x)\, \{f(y) - f(x)\}\, dy \right|,$$

and we arrive at the following estimate for all points $x \in \mathbb{R}^n$ and all numbers $0 < \varepsilon < \varepsilon_0$, namely

$$\begin{aligned}|f_\varepsilon(x) - f(x)| \le & \int\limits_{|y-x|\le\delta} K_\varepsilon(y-x)\,|f(y)-f(x)|\,dy \\ & + \int\limits_{|y-x|\ge\delta} K_\varepsilon(y-x)\,\{|f(y)|+|f(x)|\}\,dy \\ \le\; & \eta + 2\sup_{y\in\mathbb{R}^n}|f(y)| \int\limits_{|y-x|\ge\delta} K_\varepsilon(y-x)\,dy \le 2\eta.\end{aligned}$$

We summarize our considerations to

$$\sup_{x\in\mathbb{R}^n} |f_\varepsilon(x) - f(x)| \longrightarrow 0 \qquad \text{for} \quad \varepsilon \to 0+\,.$$

q.e.d.

In the sequel, we need

Proposition 3. (Partial integration in $\mathbb{R}^n$)
When the functions $f(x) \in C_0^1(\mathbb{R}^n)$ and $g(x) \in C^1(\mathbb{R}^n)$ are given, we infer

$$\int\limits_{\mathbb{R}^n} g(x)\frac{\partial}{\partial x_i} f(x)\,dx = -\int\limits_{\mathbb{R}^n} f(x)\frac{\partial}{\partial x_i} g(x)\,dx \quad \textit{for} \quad i = 1,\ldots,n.$$

Proof: On account of the property $f(x) \in C_0^1(\mathbb{R}^n)$, we find a radius $r > 0$ such that $f(x) = 0$ and $f(x)g(x) = 0$ is correct for all points $x \in \mathbb{R}^n$ with $|x_j| \ge r$ for one index $j \in \{1,\ldots,n\}$ at least. The fundamental theorem of differential- and integral-calculus yields

$$\begin{aligned}&\int\limits_{\mathbb{R}^n} \frac{\partial}{\partial x_i}\Big\{f(x)g(x)\Big\}\,dx \\ &= \int\limits_{-r}^{+r}\cdots\int\limits_{-r}^{+r}\left(\int\limits_{-r}^{+r}\frac{\partial}{\partial x_i}\Big\{f(x)g(x)\Big\}\,dx_i\right)dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n = 0.\end{aligned}$$

This implies

$$0 = \int\limits_{\mathbb{R}^n}\frac{\partial}{\partial x_i}\Big\{f(x)g(x)\Big\}\,dx = \int\limits_{\mathbb{R}^n} g(x)\frac{\partial}{\partial x_i}f(x)\,dx + \int\limits_{\mathbb{R}^n} f(x)\frac{\partial}{\partial x_i}g(x)\,dx.$$

q.e.d.

Proposition 4. *Let the function $f(x) \in C_0^k(\mathbb{R}^n, \mathbb{C})$ with $k \in \mathbb{N}_0$ be given. Then we have a sequence of polynomials with complex coefficients*

$$p_m(x) = \sum_{j_1,\ldots,j_n=0}^{N(m)} c^{(m)}_{j_1\ldots j_n} x_1^{j_1}\ldots x_n^{j_n} \quad \textit{for} \quad m=1,2,\ldots$$

such that the limit relations

$$D^\alpha p_m(x) \longrightarrow D^\alpha f(x) \qquad \textit{for} \quad m\to\infty, \quad |\alpha|\le k$$

are satisfied uniformly in each ball $B_R := \{x\in\mathbb{R}^n : |x|\le R\}$ *with the radius* $0<R<+\infty$. *Here we define the differential operator* D^α *with* $\alpha = (\alpha_1,\ldots,\alpha_n)$ *by*

$$D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}, \qquad |\alpha| := \alpha_1+\ldots+\alpha_n\,,$$

where $\alpha_1,\ldots,\alpha_n\ge 0$ *represent nonnegative integers.*

Proof: We differentiate the function $f_\varepsilon(x)$ with respect to the variables x_i, and together with Proposition 3 we see

$$\begin{aligned}\frac{\partial}{\partial x_i} f_\varepsilon(x) &= \int\limits_{\mathbb{R}^n} \left\{\frac{\partial}{\partial x_i} K_\varepsilon(y-x)\right\} f(y)\,dy\\ &= -\int\limits_{\mathbb{R}^n} \left\{\frac{\partial}{\partial y_i} K_\varepsilon(y-x)\right\} f(y)\,dy\\ &= \int\limits_{\mathbb{R}^n} K_\varepsilon(y-x)\frac{\partial}{\partial y_i} f(y)\,dy\end{aligned}$$

for $i=1,\ldots,n$. By repeated application of this device, we arrive at

$$D^\alpha f_\varepsilon(x) = \int\limits_{\mathbb{R}^n} K_\varepsilon(y-x) D^\alpha f(y)\,dy, \quad |\alpha|\le k.$$

Here we note that $D^\alpha f(y)\in C_0^0(\mathbb{R}^n)$ holds true. Due to Proposition 2, the family of functions $D^\alpha f_\varepsilon(x)$ converges uniformly on the space $\mathbb{R}^n$ towards $D^\alpha f(x)$ - for all $|\alpha|\le k$ - when $\varepsilon\to 0+$ holds true. Now we choose the radius $R>0$ such that $\operatorname{supp} f\subset B_R$ is valid. Taking the number $\varepsilon>0$ as fixed, we consider the power series

$$K_\varepsilon(z) = \frac{1}{\sqrt{\pi\varepsilon}^{\,n}}\exp\left(-\frac{|z|^2}{\varepsilon}\right) = \frac{1}{\sqrt{\pi\varepsilon}^{\,n}}\sum_{j=0}^\infty \frac{1}{j!}\left(-\frac{|z|^2}{\varepsilon}\right)^j \quad,$$

which converges uniformly in B_{2R}. Therefore, each number $\varepsilon>0$ possesses an index $N_0 = N_0(\varepsilon,R)$ such that the polynomial

$$P_{\varepsilon,R}(z) := \frac{1}{\sqrt{\pi\varepsilon}^{\,n}}\sum_{j=0}^{N_0(\varepsilon,R)} \frac{1}{j!}\left(-\frac{z_1^2+\ldots+z_n^2}{\varepsilon}\right)^j$$

is subject to the following estimate:

$$\sup_{|z|\le 2R} |K_\varepsilon(z) - P_{\varepsilon,R}(z)| \le \varepsilon.$$

With the expression

$$\widetilde{f}_{\varepsilon,R}(x) := \int_{\mathbb{R}^n} P_{\varepsilon,R}(y-x) f(y)\,dy$$

we obtain a polynomial in the variables $x_1, \ldots, x_n$ - for each $\varepsilon > 0$. Furthermore, we deduce

$$D^\alpha \widetilde{f}_{\varepsilon,R}(x) = \int_{\mathbb{R}^n} P_{\varepsilon,R}(y-x) D^\alpha f(y)\,dy \quad \text{for all} \quad x \in \mathbb{R}^n, \quad |\alpha| \le k.$$

Now we arrive at the subsequent estimate for all $|\alpha| \le k$ and $|x| \le R$, namely

$$\begin{aligned} |D^\alpha f_\varepsilon(x) - D^\alpha \widetilde{f}_{\varepsilon,R}(x)| &= \Big| \int_{|y|\le R} \Big\{ K_\varepsilon(y-x) - P_{\varepsilon,R}(y-x) \Big\} D^\alpha f(y)\,dy \Big| \\ &\le \int_{|y|\le R} |K_\varepsilon(y-x) - P_{\varepsilon,R}(y-x)||D^\alpha f(y)|\,dy \\ &\le \varepsilon \int_{|y|\le R} |D^\alpha f(y)|\,dy. \end{aligned}$$

Therefore, the polynomials $D^\alpha \widetilde{f}_{\varepsilon,R}(x)$ converge uniformly on B_R towards the derivatives $D^\alpha f(x)$. Choosing the null-sequence $\varepsilon = \frac{1}{m}$ with $m = 1, 2, \ldots$, we obtain an approximating sequence of polynomials $p_{m,R}(x) := \widetilde{f}_{\frac{1}{m},R}(x)$ in B_R, which is still dependent on the radius R. We take $r = 1, 2, \ldots$ and find polynomials $p_r = p_{m_r,r}$ satisfying

$$\sup_{x\in B_r} |D^\alpha p_r(x) - D^\alpha f(x)| \le \frac{1}{r} \quad \text{for all} \quad |\alpha| \le k \quad .$$

The sequence p_r satisfies all the properties stated above. q.e.d.

We are now prepared to prove the fundamental

Theorem 1. (The Weierstraß approximation theorem)
Let $\Omega \subset \mathbb{R}^n$ denote an open set and $f(x) \in C^k(\Omega, \mathbb{C})$ a function with the degree of regularity $k \in \mathbb{N}_0$. Then we have a sequence of polynomials with complex coefficients of the degree $N(m) \in \mathbb{N}_0$, namely

$$f_m(x) = \sum_{j_1,\ldots,j_n=0}^{N(m)} c^{(m)}_{j_1\cdots j_n} x_1^{j_1} \cdot \ldots \cdot x_n^{j_n}, \qquad x \in \mathbb{R}^n, \quad m = 1, 2, \ldots \quad ,$$