Graduate Texts in Mathematics

Joe Harris
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Moduli of Curves

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To Phil Griffiths
and David Mumford

Preface

Aims

The aim of this book is to provide a guide to a rich and fascinating subject: algebraic curves, and how they vary in families. The revolution that the field of algebraic geometry has undergone with the introduction of schemes, together with new ideas, techniques and viewpoints introduced by Mumford and others, have made it possible for us to understand the behavior of curves in ways that simply were not possible a half-century ago. This in turn has led, over the last few decades, to a burst of activity in the area, resolving long-standing problems and generating new and unforeseen results and questions. We hope to acquaint you both with these results and with the ideas that have made them possible.

The book isn't intended to be a definitive reference: the subject is developing too rapidly for that to be a feasible goal, even if we had the expertise necessary for the task. Our preference has been to focus on examples and applications rather than on foundations. When discussing techniques we've chosen to sacrifice proofs of some, even basic, results — particularly where we can provide a good reference — in order to show how the methods are used to study moduli of curves. Likewise, we often prove results in special cases which we feel bring out the important ideas with a minimum of technical complication.

Chapters 1 and 2 provide a synopsis of basic theorems and conjectures about Hilbert schemes and moduli spaces of curves, with few or no details about techniques or proofs. Use them more as a guide to the literature than as a working manual. Chapters 3 through 6 are, by contrast, considerably more self-contained and approachable. Ultimately, if you want to investigate fully any of the topics we discuss, you'll have to go beyond the material here; but you *will* learn the techniques fully enough, and see enough complete proofs, that when you finish a section here you'll be equipped to go exploring on your own.

If your goal is to work with families of curves, we'd therefore suggest that you begin by skimming the first two chapters and then tackle the later chapters in detail, referring back to the first two as necessary.

Contents

As for the contents of the book: Chapters 1 and 2 are largely expository: for the most part, we discuss in general terms the problems associated with moduli and parameter spaces of curves, what's known about them, and what sort of behavior we've come to expect from them. In Chapters 3 through 5 we develop the techniques that have allowed us to analyze moduli spaces: deformations, specializations (of curves, of maps between them and of linear series on them), tools for making a variety of global enumerative calculations, geometric invariant theory, and so on. Finally, in Chapter 6, we use the ideas and techniques introduced in preceding chapters to prove a number of basic results about the geometry of the moduli space of curves and about various related spaces.

Prerequisites

What sort of background do we expect you to have before you start reading? That depends on what you want to get out of the book. We'd hope that even if you have only a basic grounding in modern algebraic geometry and a slightly greater familiarity with the theory of a fixed algebraic curve, you could read through most of this book and get a sense of what the subject is about: what sort of questions we ask, and some of the ways we go about answering them. If your ambition is to work in this area, of course, you'll need to know more; a working knowledge with many of the topics covered in *Geometry of algebraic curves, I* [7] first and foremost. We could compile a lengthy list of other subjects with which some acquaintance would be helpful. But, instead, we encourage you to just plunge ahead and fill in the background as needed; again, we've tried to write the book in a style that makes such an approach feasible.

Navigation

In keeping with the informal aims of the book, we have used only two levels of numbering with arabic for chapters and capital letters for sections within each chapter. All labelled items in the book are numbered consecutively within each chapter: thus, the orderings of such items by label and by position in the book agree.

There is a single index. However, its first page consists of a list of symbols, giving for each a single defining occurrence. These, and other, references to symbols also appear in the main body of the index where they are alphabetized "as read": for example, references to $\overline{\mathcal{M}}_g$ will be found under `Mgbar`; to κ_i under `kappai`. Bold face entries in the main body index point to the defining occurrence of the cited term. References to all the main results stated in the book can be found under the heading `theorems`.

Production acknowledgements

This book was designed by the authors who provided Springer with the PostScript file from which the plates were produced. The type is a very slightly modified version of the Lucida font family designed by Chuck Bigelow and Kristin Holmes. (We added swashes to a few characters in the `\mathcal` alphabet to make them easier to distinguish from the corresponding upper-case `\mathit` character. These alphabets are often paired: a `\mathcal` character is used for the total space of a family and the `\mathit` version for an element.) It was coded in a customized version of the LaTeX2e format and typeset using Blue Sky Research's Textures TeX implementation with EPS figures created in Macromedia's Freehand7 illustration program.

A number of people helped us with the production of the book. First and foremost, we want to thank Greg Langmead who did a truly wonderful job of producing an initial version of both the LaTeX code and the figures from our earlier WYSIWYG drafts. Dave Bayer offered invaluable programming assistance in solving many problems. Most notably, he devoted considerable effort to developing a set of macros for overlaying text generated within TeX onto figures. These allow precise one-time text placement independent of the scale of the figure and proved invaluable both in preparing the initial figures and in solving float placement problems. If you're interested, you can obtain the macros, which work with all formats, by e-mailing Dave at `bayer@math.columbia.edu`.

Frank Ganz at Springer made a number of comments to improve the design and assisted in solving some of the formatting problems he raised. At various points, Donald Arseneau, Berthold Horn, Vincent Jalby and Sorin Popescu helped us solve or work around various difficulties. We are grateful to all of them.

Lastly, we wish to thank our patient editor, Ina Lindemann, who was never in our way but always ready to help.

Mathematical acknowledgements

You should not hope to find here the sequel to *Geometry of algebraic curves, I* [7] announced in the preface to that book. As we've already noted, our aim is far from the "comprehensive and self-contained account" which was the goal of that book, and our text lacks its uniformity. The promised second volume is in preparation by Enrico Arbarello, Maurizio Cornalba and Phil Griffiths.

A few years ago, these authors invited us to attempt to merge our then current manuscript into theirs. However, when the two sets of material were assembled, it became clear to everyone that ours was so far from meeting the standards set by the first volume that such a merger made little sense. Enrico, Maurizio and Phil then, with their usual generosity, agreed to allow us to withdraw from their project and to publish what we had written here. We cannot too strongly acknowledge our admiration for the kindness with which the partnership was proposed and the grace with which it was dissolved nor our debt to them for the influence their ideas have had on our understanding of curves and their moduli.

The book is based on notes from a course taught at Harvard in 1990, when the second author was visiting, and we'd like to thank Harvard University for providing the support to make this possible, and Fordham University for granting the second author both the leave for this visit and a sabbatical leave in 1992-93. The comments of a number of students who attended the Harvard course were very helpful to us: in particular, we thank Dan Abramovich, Jean-Francois Burnol, Lucia Caporaso and James McKernan. We owe a particular debt to Angelo Vistoli, who also sat in on the course, and patiently answered many questions about deformation theory and algebraic stacks.

There are many others as well with whom we've discussed the various topics in this book, and whose insights are represented here. In addition to those mentioned already, we thank especially David Eisenbud, Bill Fulton and David Gieseker.

We to thank Armand Brumer, Anton Dzhamay, Carel Faber, Bill Fulton, Rahul Pandharipande, Cris Poor, Sorin Popescu and Monserrat Teixidor i Bigas who volunteered to review parts of this book. Their comments enabled us to eliminate many errors and obscurities. For any that remain, the responsibility is ours alone.

Finally, we thank our respective teachers, Phil Griffiths and David Mumford. The beautiful results they proved and the encouragement they provided energized and transformed the study of algebraic curves — for us and for many others. We gratefully dedicate this book to them.

Contents

Chapter 1

Parameter spaces: constructions and examples

A Parameters and moduli

Before we take up any of the constructions that will occupy us in this chapter, we want to make a few general remarks about moduli problems in general.

What is a moduli problem? Typically, it consists of two things. First of all, we specify a class of objects (which could be schemes, sheaves, morphisms or combinations of these), together with a notion of what it means to have a family of these objects over a scheme B. Second, we choose a (possibly trivial) equivalence relation $\sim$ on the set $S(B)$ of all such families over each B. We use the rather vague term "object" deliberately because the possibilities we have in mind are wide-ranging. For example, we might take our families to be

1. smooth flat morphisms $C \twoheadrightarrow B$ whose fibers are smooth curves of genus g, or

2. subschemes C in $\mathbb{P}^r \times B$, flat over B, whose fibers over B are curves of fixed genus g and degree d,

and so on. We can loosely consider the elements of $S(\mathrm{Spec}(\mathbf{C}))$ as the objects of our moduli problem and the elements of $S(B)$ over other bases as families of such objects parameterized by the complex points of B.[1]

The equivalence relations we will wish to consider will vary considerably even for a fixed class of objects: in the second case cited above, we might wish to consider two families equivalent if

[1]More generally, we may consider elements of $S(\mathrm{Spec}(k))$ for any field k as objects of our moduli problem defined over k.

1. the two subschemes of $\mathbb{P}^r \times B$ are equal,
2. the two subcurves are projectively equivalent over B, or
3. the two curves are (biregularly) isomorphic over B.

In any case, we build a functor $\mathbf{F}$ from the category of schemes to that of sets by the rule

$$\mathbf{F}(B) = S(B)/\sim$$

and call $\mathbf{F}$ the moduli functor of our moduli problem.

The fundamental first question to answer in studying a given moduli problem is: to what extent is the functor $\mathbf{F}$ representable? Recall that $\mathbf{F}$ is *representable* in the category of schemes if there is a scheme $\mathcal{M}$ and an isomorphism Ψ (of functors from schemes to sets) between $\mathbf{F}$ and the *functor of points* of $\mathcal{M}$. This last is the functor $\mathrm{Mor}_{\mathcal{M}}$ whose value on B is the set $\mathrm{Mor}_{\mathrm{sch}}(B, \mathcal{M})$ of all morphisms of schemes from B to $\mathcal{M}$.

Definition (1.1) *If* $\mathbf{F}$ *is representable by* $\mathcal{M}$*, then we say that the scheme* $\mathcal{M}$ *is a* fine moduli space *for the moduli problem* $\mathbf{F}$.

Representability has a number of happy consequences for the study of $\mathbf{F}$. If $\varphi : \mathcal{D} \to B$ is any family in (i.e., any element of) $S(B)$, then $\chi = \Psi(\varphi)$ is a morphism from B to $\mathcal{M}$. Intuitively, (closed) points of $\mathcal{M}$ classify the objects of our moduli problem and the map χ sends a (closed) point b of B to the moduli point in $\mathcal{M}$ determined by the fiber $\mathcal{D}_b$ of $\mathcal{D}$ over b. Going the other way, pulling back the identity map of $\mathcal{M}$ itself via Ψ constructs a family $\mathbf{1} : \mathcal{C} \to \mathcal{M}$ in $S(\mathcal{M})$ called the *universal* family. The reason for this name is that, given any morphism $\chi : B \to \mathcal{M}$ defined as above, there is a commutative fiber-product diagram

(1.2)
$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{C} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\mathbf{1}} \\ B & \xrightarrow{\ \chi\ } & \mathcal{M} \end{array}$$

with $\varphi : \mathcal{D} \to B$ in $S(B)$ and $\Psi(\varphi) = \chi$. In sum, every family over B is the pullback of $\mathcal{C}$ via a *unique* map of B to $\mathcal{M}$ and we have a perfect dictionary enabling us to translate between information about the geometry of families of our moduli problem and information about the geometry of the moduli space $\mathcal{M}$ itself. One of the main themes of moduli theory is to bring information about the objects of our moduli problem to bear on the study of families and vice versa: the dictionary above is a powerful tool for relating these two types of information.

Unfortunately, few natural moduli functors are representable by schemes: we'll look at the reasons for this failure in the next chapter. One response to this failure is to look for a larger category (e.g., algebraic spaces, algebraic stacks, ...) in which F can be represented: the investigation of this avenue will also be postponed until the next chapter. Here we wish to glance briefly at a second strategy: to find a scheme $\mathcal{M}$ that captures enough of the information in the functor F to provide us with a "concise edition" of the dictionary above.

The standard way to do this is to ask only for a natural transformation of functors $\Psi = \Psi_{\mathcal{M}}$ from F to $\mathrm{Mor}(\cdot, \mathcal{M})$ rather than an isomorphism. Then, for each family $\varphi : \mathcal{D} \to B$ in $S(B)$, we still have a morphism $\chi = \Psi(\varphi) : B \to \mathcal{M}$ as above. Moreover, these maps are still natural in that, if $\varphi' : \mathcal{D}' = \mathcal{D} \times_B B' \to B'$ is the base change by a map $\xi : B' \to B$, then $\chi' = \Psi(\varphi') = \Psi(\varphi) \circ \xi$. This requirement, however, is far from determining $\mathcal{M}$. Indeed, given any solution $(\mathcal{M}, \Psi)$ and any morphism $\pi : \mathcal{M} \to \mathcal{M}'$, we get another solution $(\mathcal{M}', \pi \circ \Psi)$. For example, we could *always* take $\mathcal{M}'$ to equal $\mathrm{Spec}(\mathbf{C})$ and $\Psi(\varphi)$ to be the unique morphism $B \to \mathrm{Spec}(\mathbf{C})$ and then our dictionary would have only blank pages; or, we could take the disjoint union of the "right" $\mathcal{M}$ with any other scheme. We can rule such cases out by requiring that the complex points of $\mathcal{M}$ correspond bijectively to the objects of our moduli problem. This still doesn't fix the scheme structure on $\mathcal{M}$: it leaves us the freedom to compose, as above, with a map $\pi : \mathcal{M} \to \mathcal{M}'$ as long as π itself is bijective on complex points. For example, we would certainly want the moduli space $\mathcal{M}$ of lines through the origin in $\mathbf{C}^2$ to be $\mathbf{P}^1$ but our requirements so far don't exclude the possibility of taking instead the cuspidal rational curve $\mathcal{M}'$ with equation $y^2z = x^3$ in $\mathbf{P}^2$ which is the image of $\mathbf{P}^1$ under the map $[a, b] \to [a^2b, a^3, b^3]$. This pathology can be eliminated by requiring that $\mathcal{M}$ be universal with respect to the existence of the natural transformation Ψ: cf. the first exercise below. When all this holds, we say that $(\mathcal{M}, \Psi)$, or more frequently $\mathcal{M}$, is a *coarse moduli space* for the functor F. Formally,

Definition (1.3) *A scheme $\mathcal{M}$ and a natural transformation $\Psi_{\mathcal{M}}$ from the functor* F *to the functor of points* $\mathrm{Mor}_{\mathcal{M}}$ *of $\mathcal{M}$ are a* coarse moduli space *for the functor* F *if*

1) *The map $\Psi_{\mathrm{Spec}(\mathbf{C})} : \mathrm{F}(\mathrm{Spec}(\mathbf{C})) \to \mathcal{M}(\mathbf{C}) = \mathrm{Mor}(\mathrm{Spec}(\mathbf{C}), \mathcal{M})$ is a set bijection.*[2]

2) *Given another scheme $\mathcal{M}'$ and a natural transformation $\Psi_{\mathcal{M}'}$ from $\mathrm{F} \to \mathrm{Mor}_{\mathcal{M}'}$, there is a unique morphism $\pi : \mathcal{M} \to \mathcal{M}'$ such that*

[2]Or more generally require this with **C** replaced by any algebraically closed field.