

DIFFUSIONS, MARKOV PROCESSES AND MARTINGALES

Volume 1
FOUNDATIONS

扩散 马尔可夫过程和鞅
第 1 卷

L. C. G. Rogers & D. Williams

Cambridge Mathematical Library

世界图书出版公司

Diffusions, Markov Processes, and Martingales

Volume 1: FOUNDATIONS

2nd Edition

L. C. G. ROGERS

*School of Mathematical Sciences,
University of Bath*

and

DAVID WILLIAMS

*Department of Mathematics,
University of Wales, Swansea*



CAMBRIDGE
UNIVERSITY PRESS

书 名: Diffusions, Markov Processes and Martingales Vol. 1
作 者: L.C.G.Rogers, D.Williams
中译名: 扩散 马尔可夫过程和鞅 第1卷
出 版 者: 世界图书出版公司北京公司
印 刷 者: 北京世图印刷厂
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)
开 本: 24 印 张: 17
出版年代: 2003 年 1 月
书 号: 7-5062-5921-4/O · 345
版权登记: 图字: 01-2002-5626
定 价: 63.00 元

世界图书出版公司北京公司已获得 Cambridge 出版社授权在中国大陆独家重印发行。

Diffusions, Markov Processes,
and Martingales

Volume 1: Foundations

2nd Edition

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS

The Edinburgh Building, Cambridge CB2 2RU, UK <http://www.cup.cam.ac.uk>
40 West 20th Street, New York, NY 10011-4211, USA <http://www.cup.org>
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

© John Wiley & Sons Ltd 1979, 1994

© Cambridge University Press 2000

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 1979 by John Wiley & Sons Ltd, Chichester

Second edition 1994 published by John Wiley & Sons Ltd

reissued by Cambridge University Press 2000

A catalogue record for this book is available from the British Library

ISBN 0 521 77594 9 paperback

This edition of *Diffusions, Markov Processes and
Martingales Vol. 1* by L.C.G.Rogers and D.Williams
is published by arrangement with the Syndicate of
the Press of University of Cambridge, Cambridge,
England.

Licensed edition for sale in the People's Republic of
China only. Not for export elsewhere.

For our parents

From the Original (1979)

Preface

Long ago (or so it seems today), Chung wrote on page 196 of his book [1]: 'One wonders if the present theory of stochastic processes is not still too difficult for applications.' Advances in the theory since that time have been phenomenal, but these have been accompanied by an increase in the technical difficulty of the subject so bewildering as to give a quaint charm to Chung's use of the word 'still'. Meyer writes in the preface to his definitive account of stochastic integral theory: '*... il faut... un cours de six mois sur les définitions. Que peut on y faire?*'

I have thought up as intuitive a picture of the subject as I can, written it down at speed, and refused to be lured back by piety (or even by wit!) to cancel half a line. 'First' intuition, which is what you need when you are learning the subject, is raw, rough and ready; and, as you have guessed, I make the excuse that it demands a compatible style and lack of polish.

Note that I wrote '*first intuition*'. Consider an example. Meyer's concept of a *right process* is exactly right for Markov process theory, but the concept is the result of a long evolution. To understand it properly, you need a highly developed intuition, and that takes time to acquire. The difficulty with the best advanced literature is that its authors have too *much* intuition; never make the mistake of thinking otherwise.

My aim then is to sharpen your intuition to a point where the advanced abstract literature becomes accessible, enjoyable and 'relevant'. Like my expository article [1], this is a missionary tract not a theological treatise. (Those of you who have read my article [1] will see that this book often follows it very closely, except that now I have the time and the duty to be more obviously appreciative of the abstract theory!)

I believe that, in the end, it is *applications* which justify mathematics. The 'artistic' justification of pure mathematics in terms of intrinsic qualities like elegance and generality rings rather hollow in my ears when I compare the best mathematics with the greatest music. Many applied workers will regard this book as extremely 'pure', but I see it as *one stage in shunting pure theory over towards applications*. The shunting is not always necessary: time and again, one finds 'applied' papers which 'solve' problems long since solved for 'purely

theoretical' purposes. Moral: the pure/applied division of probability theory (as of mathematics in general) is a nonsense.

Acknowledgements. This is an appropriate place at which to thank David Kendall and Harry Reuter for teaching me probability theory and for giving me an enthusiasm for the subject which is wearing well. My best way to thank them is to try to share that enthusiasm.

I have to say another huge 'thank you' to David Kendall for the immense amount of work he has done in making editorial comments on the original manuscript. I now see that my determination to convey a sense of adventure did need to be tempered by a greater concern for the reader's sense of security. So I have acceded to many of David Kendall's requests for 'more details'; and as a result, you will learn more techniques of calculation and have a clearer idea of several concepts. (But I still see it as part of my job to keep you on your toes!)

I am very grateful to Ronald Getoor and André Meyer for clearing up some confusions.

I have been extremely fortunate in having been able to rely on the superb typing skills of Sheila Campbell, Eileen Jenkins and Gladys Maddocks; my thanks and best wishes to them.

I thank Springer-Verlag and the authors for granting me permission to quote from Chung [1] in Section III.44, from Getoor [1] in Section III.54, and from Chung [1] and Meyer [1] earlier in this preface.

Finally, I have to thank James Cameron and Wiley for encouragement and great patience; and subeditors, copy-editors, and printers, whose skills have much impressed me.

David Williams
Swansea, 1978

Preface to the Second Edition

This second edition differs profoundly from the first—and not only in having two authors rather than one. We retain the Gallic tradition of dividing the volume into three massive chapters: Chapter I, which says why the subject is worth studying; Chapter II, which provides background; and Chapter III, which presents an account of Markov processes. Chapter I is now much more extensive and wide-ranging, and covers much work done since the first edition appeared. Chapter II is now a highly systematic account, with detailed proofs, of what every young probabilist must know. It is rather unashamedly a sequel to DW's *Probability with Martingales*, Cambridge University Press having been very generous in allowing us to follow that account closely (but without many proofs, without the examples, etc.). *It is perfectly possible to read Chapter II before Chapter I if you so wish.* We would suggest however that you try things in the order 'heuristics then rigour':

*'Our doubts are traitors,
And make us lose the good we oft might win,
Through fearing to attempt.*

(W. Shakespeare, *Measure for Measure*.)

Chapter III seems to have been regarded as the most successful part of the original; and it is reproduced here without much modification (except that some of the functional analysis is given fuller treatment). It was always intended as a missionary tract on Markov processes. The full theory may be found in Sharpe [1] and in the final two volumes of the probabilist's bible, Dellacherie and Meyer [1]. All kinds of important developments are ignored in Chapter III: they would require another complete volume, and will be, or are, covered by greater experts. Dawson's eagerly awaited treatment [1] of measure-valued processes has now appeared; Mark Davis has a very nice new book [4] on piecewise-deterministic Markov processes; and so on. You can access the huge literature on measure-valued processes via Dawson's account.

The musical allusions in the first edition have been excised. Apparently many people found them annoying. 'Would David Williams like a book on mathematics

filled with references to baseball?', they say. (To which the answer is, of course, 'Yes.') So, this is Mathematics all the way from A to Zzzz—or from Ω on, if you want to be rigorous.

Our thanks to Sue Collins and Wolfgang Stummer, and to other colleagues at Bath, Cambridge, and Queen Mary and Westfield College, London. Our thanks too to Helen Ramsey and other Wiley staff for suggesting this new version; and the copy-editor and printer whose skills have impressed us.

Chris Rogers
David Williams
November 1993

Some Frequently Used Notation

We use ‘:=’ to mean ‘is defined to equal’. This Pascal notation can also be used in reverse. We define

$$\mathbb{Z}^+ := \{0, 1, 2, \dots\} \supseteq \{1, 2, 3, \dots\} =: \mathbb{N},$$

$$\mathbb{R}^+ := [0, \infty), \quad \mathbb{R}^{++} := (0, \infty), \quad \mathbb{Q}^+ := \mathbb{Q} \cap \mathbb{R}^+.$$

We neaten layout, and make things easier for our printers, by the use of alternative notations:

$X(t_1, \omega)$ for $X_{t_1}(\omega)$, $f_{n(t)}$ for f_{n_1} , $\mathcal{F}(T_1)$ for \mathcal{F}_{T_1} , $P_t f(x)$ for $(P_t f)(x)$, etc. Once things are underway, such switches in notation will be made without comment. The composition notation

$$f \circ g(t) := f(g(t))$$

will often be used for tidiness.

If f and g are real numbers or real-valued functions, we define

$$f \vee g := \max(f, g), \quad f \wedge g := \min(f, g), \quad f^+ := f \vee 0, \quad f^- := (-f) \vee 0;$$

hence $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

If \mathcal{H} is a set of real-valued functions, we write

\mathcal{H}^+ for the set of non-negative elements of \mathcal{H} ,

$b\mathcal{H}^+$ for the set of bounded elements in \mathcal{H} .

If Σ is a σ -algebra, we write

$m\Sigma$ for the set of real-valued (or perhaps $[\infty, \infty]$ -valued)

Σ -measurable functions,

$b\Sigma$ for the space of bounded Σ -measurable functions.

If S is a topological space, we write

$C(S)$ for the space of all continuous functions from S to \mathbb{R} .

$C_b(S)$ for the space of all bounded continuous functions from S to \mathbb{R} .

Monotone convergence. We write ' $s \uparrow t$ ' to signify that $s \rightarrow t, s \leq t$; and ' $s \uparrow \uparrow t$ ' to signify that $s \rightarrow t, s < t$. If (s_n) is a sequence then ' $s_n \uparrow t$ ' signifies that $s_n \rightarrow t, s_n \leq s_{n+1} \leq t$; while ' $s_n \uparrow \uparrow t$ ' signifies that $s_n \rightarrow t, s_n \leq s_{n+1} < t$. If f_n and f are real-valued functions then (for example) $f: \uparrow \lim f_n$ signifies that $f_n \uparrow f$ pointwise.

Contents

Some Frequently Used Notation

xix

CHAPTER I. BROWNIAN MOTION

1. INTRODUCTION	1
1. What is Brownian motion, and why study it?	1
2. Brownian motion as a martingale	2
3. Brownian motion as a Gaussian process	3
4. Brownian motion as a Markov process	5
5. Brownian motion as a diffusion (and martingale)	7
2. BASICS ABOUT BROWNIAN MOTION	10
6. Existence and uniqueness of Brownian motion	10
7. Skorokhod embedding	13
8. Donsker's Invariance Principle	16
9. Exponential martingales and first-passage distributions	18
10. Some sample-path properties	19
11. Quadratic variation	21
12. The strong Markov property	21
13. Reflection	25
14. Reflecting Brownian motion and local time	27
15. Kolmogorov's test	31
16. Brownian exponential martingales and the Law of the Iterated Logarithm	31
3. BROWNIAN MOTION IN HIGHER DIMENSIONS	36
17. Some martingales for Brownian motion	36
18. Recurrence and transience in higher dimensions	38
19. Some applications of Brownian motion to complex analysis	39
20. Windings of planar Brownian motion	43
21. Multiple points, cone points, cut points	45

22.	Potential theory of Brownian motion in \mathbb{R}^d ($d \geq 3$)	46
23.	Brownian motion and physical diffusion	51
4.	GAUSSIAN PROCESSES AND LÉVY PROCESSES	55
	<i>Gaussian processes</i>	
24.	Existence results for Gaussian processes	55
25.	Continuity results	59
26.	Isotropic random flows	66
27.	Dynkin's Isomorphism Theorem	71
	<i>Lévy processes</i>	
28.	Lévy processes	73
29.	Fluctuation theory and Wiener-Hopf factorisation	80
30.	Local time of Lévy processes	82
	CHAPTER II. SOME CLASSICAL THEORY	
1.	BASIC MEASURE THEORY	85
	<i>Measurability and measure</i>	
1.	Measurable spaces; σ -algebras; π -systems; d -systems	85
2.	Measurable functions	88
3.	Monotone-Class Theorems	90
4.	Measures; the uniqueness lemma; almost everywhere; a.e. (μ, Σ)	91
5.	Carathéodory's Extension Theorem	93
6.	Inner and outer μ -measures; completion	94
	<i>Integration</i>	
7.	Definition of the integral $\int f d\mu$	95
8.	Convergence theorems	96
9.	The Radon-Nikodým Theorem; absolute continuity; $\lambda \ll \mu$ notation; equivalent measures	98
10.	Inequalities; \mathcal{L}^p and L^p spaces ($p \geq 1$)	99
	<i>Product structures</i>	
11.	Product σ -algebras	101
12.	Product measure; Fubini's Theorem	102
13.	Exercises	104
2.	BASIC PROBABILITY THEORY	108
	<i>Probability and expectation</i>	
14.	Probability triple; almost surely (a.s.); a.s. (\mathbf{P}) , a.s. $(\mathbf{P}, \mathcal{F})$	108

15.	$\limsup E_n$; First Borel–Cantelli Lemma	109
16.	Law of random variable; distribution function; joint law	110
17.	Expectation; $\mathbf{E}(X; F)$	110
18.	Inequalities: Markov, Jensen, Schwarz, Tchebychev	111
19.	Modes of convergence of random variables	113
	<i>Uniform integrability and \mathcal{L}^1 convergence</i>	
20.	Uniform integrability	114
21.	\mathcal{L}^1 convergence	115
	<i>Independence</i>	
22.	Independence of σ -algebras and of random variables	116
23.	Existence of families of independent variables	118
24.	Exercises	119
3.	STOCHASTIC PROCESSES	119
	<i>The Daniell–Kolmogorov Theorem</i>	
25.	(E^T, \mathcal{E}^T) ; σ -algebras on function space; cylinders and σ -cylinders	119
26.	Infinite products of probability triples	121
27.	Stochastic process; sample function; law	121
28.	Canonical process	122
29.	Finite-dimensional distributions; sufficiency; compatibility	123
30.	The Daniell–Kolmogorov (DK) Theorem: ‘compact metrizable’ case	124
31.	The Daniell–Kolmogorov (DK) Theorem: general case	126
32.	Gaussian processes; pre-Brownian motion	127
33.	Pre-Poisson set functions	128
	<i>Beyond the DK Theorem</i>	
34.	Limitations of the DK Theorem	128
35.	The role of outer measures	129
36.	Modifications; indistinguishability	130
37.	Direct construction of Poisson measures and subordinators, and of local time from the zero set; Azéma’s martingale	131
38.	Exercises	136
4.	DISCRETE-PARAMETER MARTINGALE THEORY	137
	<i>Conditional expectation</i>	
30.	Fundamental theorem and definition	137
40.	Notation; agreement with elementary usage	138
41.	Properties of conditional expectation: a list	139
42.	The role of versions; regular conditional probabilities and pdfs	140

43.	A counterexample	141
44.	A uniform-integrability property of conditional expectations	142
	<i>(Discrete-parameter) martingales and supermartingales</i>	
45.	Filtration; filtered space; adapted process; natural filtration	143
46.	Martingale; supermartingale; submartingale	144
47.	Previsible process; gambling strategy; a fundamental principle	144
48.	Doob's Upcrossing Lemma	145
49.	Doob's Supermartingale-Convergence Theorem	146
50.	\mathcal{L}^1 convergence and the UI property	147
51.	The Lévy-Doob Downward Theorem	148
52.	Doob's Submartingale and \mathcal{L}^p Inequalities	150
53.	Martingales in \mathcal{L}^2 ; orthogonality of increments	152
54.	Doob decomposition	153
55.	The $\langle M \rangle$ and $[M]$ processes	154
	<i>Stopping times, optional stopping and optional sampling</i>	
56.	Stopping time	155
57.	Optional-stopping theorems	156
58.	The pre- T σ -algebra \mathcal{F}_T	158
59.	Optional sampling	159
60.	Exercises	161
5.	CONTINUOUS-PARAMETER SUPERMARTINGALES	163
	<i>Regularisation: R-supermartingales</i>	
61.	Orientation	163
62.	Some real-variable results	163
63.	Filtrations; supermartingales; R-processes, R-supermartingales	166
64.	Some important examples	167
65.	Doob's Regularity Theorem: Part I	169
66.	Partial augmentation	171
67.	Usual conditions; R-filtered space; usual augmentation; R-regularisation	172
68.	A necessary pause for thought	174
69.	Convergence theorems for R-supermartingales	175
70.	Inequalities and \mathcal{L}^p convergence for R-submartingales	177
71.	Martingale proof of Wiener's Theorem; canonical Brownian motion	178
72.	Brownian motion relative to a filtered space	180
	<i>Stopping times</i>	
73.	Stopping time T ; pre- T σ -algebra \mathcal{G}_T ; progressive process	181
74.	First-entrance (début) times; hitting times; first-approach times: the easy cases	183

75. Why 'completion' in the usual conditions has to be introduced	184
76. Début and Section Theorems	186
77. Optional Sampling for R-supermartingales under the usual conditions	188
78. Two important results for Markov-process theory	191
79. Exercises	192
6. PROBABILITY MEASURES ON LUSIN SPACES	200
' <i>Weak convergence</i> '	
80. $C(J)$ and $\Pr(J)$ when J is compact Hausdorff	202
81. $C(J)$ and $\Pr(J)$ when J is compact metrizable	203
82. Polish and Lusin spaces	205
83. The $C_b(S)$ topology of $\Pr(S)$ when S is a Lusin space; Prohorov's Theorem	207
84. Some useful convergence results	211
85. Tightness in $\Pr(W)$ when W is the path-space $W := C([0, \infty); \mathbb{R})$	213
86. The Skorokhod representation of $C_b(S)$ convergence on $\Pr(S)$	215
87. Weak convergence versus convergence of finite-dimensional distributions	216
 ' <i>Regular conditional probabilities</i> '	
88. Some preliminaries	217
89. The main existence theorem	218
90. Canonical Brownian Motion $\text{CBM}(\mathbb{R}^N)$; Markov property of \mathbf{P}^x laws	220
91. Exercises	222

CHAPTER III. MARKOV PROCESSES

1. TRANSITION FUNCTIONS AND RESOLVENTS	227
1. What is a (continuous-time) Markov process?	227
2. The finite-state-space Markov chain	228
3. Transition functions and their resolvents	231
4. Contraction semigroups on Banach spaces	234
5. The Hille–Yosida Theorem	237
2. FELLER–DYNKIN PROCESSES	240
6. Feller–Dynkin (FD) semigroups	240
7. The existence theorem: canonical FD processes	243
8. Strong Markov property: preliminary version	247
9. Strong Markov property: full version; Blumenthal's 0–1 Law	249