Graduate Texts in Mathematics

Jürgen Jost

Partial Differential Equations

Second Edition

偏微分方程 第2版

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Preface

This textbook is intended for students who wish to obtain an introduction to the theory of partial differential equations (PDEs, for short), in particular, those of elliptic type. Thus, it does not offer a comprehensive overview of the whole field of PDEs, but tries to lead the reader to the most important methods and central results in the case of elliptic PDEs. The guiding question is how one can find a solution of such a PDE. Such a solution will, of course, depend on given constraints and, in turn, if the constraints are of the appropriate type, be uniquely determined by them. We shall pursue a number of strategies for finding a solution of a PDE; they can be informally characterized as follows:

(0) *Write down an* **explicit formula** *for the solution in terms of the given data (constraints).*
This may seem like the best and most natural approach, but this is possible only in rather particular and special cases. Also, such a formula may be rather complicated, so that it is not very helpful for detecting qualitative properties of a solution. Therefore, mathematical analysis has developed other, more powerful, approaches.

(1) *Solve a sequence of auxiliary problems that* **approximate** *the given one, and show that their solutions converge to a solution of that original problem.*
Differential equations are posed in spaces of functions, and those spaces are of infinite dimension. The strength of this strategy lies in carefully choosing finite-dimensional approximating problems that can be solved explicitly or numerically and that still share important crucial features with the original problem. Those features will allow us to control their solutions and to show their convergence.

(2) *Start anywhere, with the required constraints satisfied, and let things* **flow** *toward a solution.*
This is the diffusion method. It depends on characterizing a solution of the PDE under consideration as an asymptotic equilibrium state for a diffusion process. That diffusion process itself follows a PDE, with an additional independent variable. Thus, we are solving a PDE that is more complicated than the original one. The advantage lies in the fact that we can simply start anywhere and let the PDE control the evolution.

(3) *Solve an* **optimization** *problem, and identify an optimal state as a solution of the PDE.*
This is a powerful method for a large class of elliptic PDEs, namely, for those that characterize the optima of variational problems. In fact, in applications in physics, engineering, or economics, most PDEs arise from such optimization problems. The method depends on two principles. First, one can demonstrate the existence of an optimal state for a variational problem under rather general conditions. Second, the optimality of a state is a powerful property that entails many detailed features: If the state is not very good at every point, it could be improved and therefore could not be optimal.
(4) **Connect** *what you want to know to what you know already.*
This is the continuity method. The idea is that, if you can connect your given problem continuously with another, simpler, problem that you can already solve, then you can also solve the former. Of course, the continuation of solutions requires careful control.

The various existence schemes will lead us to another, more technical, but equally important, question, namely, the one about the regularity of solutions of PDEs. If one writes down a differential equation for some function, then one might be inclined to assume explicitly or implicitly that a solution satisfies appropriate differentiability properties so that the equation is meaningful. The problem, however, with many of the existence schemes described above is that they often only yield a solution in some function space that is so large that it also contains nonsmooth and perhaps even noncontinuous functions. The notion of a solution thus has to be interpreted in some generalized sense. It is the task of regularity theory to show that the equation in question forces a generalized solution to be smooth after all, thus closing the circle. This will be the second guiding problem of the present book.

The existence and the regularity questions are often closely intertwined. Regularity is often demonstrated by deriving explicit estimates in terms of the given constraints that any solution has to satisfy, and these estimates in turn can be used for compactness arguments in existence schemes. Such estimates can also often be used to show the uniqueness of solutions, and of course, the problem of uniqueness is also fundamental in the theory of PDEs.

After this informal discussion, let us now describe the contents of this book in more specific detail.

Our starting point is the Laplace equation, whose solutions are the harmonic functions. The field of elliptic PDEs is then naturally explored as a generalization of the Laplace equation, and we emphasize various aspects on the way. We shall develop a multitude of different approaches, which in turn will also shed new light on our initial Laplace equation. One of the important approaches is the heat equation method, where solutions of elliptic PDEs are obtained as asymptotic equilibria of parabolic PDEs. In this sense, one chapter treats the heat equation, so that the present textbook definitely is not confined to elliptic equations only. We shall also treat the wave equation as the prototype of a hyperbolic PDE and discuss its relation to the Laplace and heat equations. In the context of the heat equation, another chapter develops the theory of semigroups and explains the connection with Brownian motion.

Other methods for obtaining the existence of solutions of elliptic PDEs, like the difference method, which is important for the numerical construction of solutions; the Perron method; and the alternating method of H.A. Schwarz; are based on the maximum principle. We shall present several versions of the maximum principle that are also relevant for applications to nonlinear PDEs.

In any case, it is an important guiding principle of this textbook to develop methods that are also useful for the study of nonlinear equations, as those present the research perspective of the future. Most of the PDEs occurring in applications in the sciences, economics, and engineering are of nonlinear types. One should keep in mind, however, that, because of the multitude of occurring equations and resulting phenomena, there cannot exist a unified theory of nonlinear (elliptic) PDEs, in contrast to the linear case. Thus, there are also no universally applicable methods, and we aim instead at doing justice to this multitude of phenomena by developing very diverse methods.

Thus, after the maximum principle and the heat equation, we shall encounter variational methods, whose idea is represented by the so-called Dirichlet principle. For that purpose, we shall also develop the theory of Sobolev spaces, including fundamental embedding theorems of Sobolev, Morrey, and John–Nirenberg. With the help of such results, one can show the smoothness of the so-called weak solutions obtained by the variational approach. We also treat the regularity theory of the so-called strong solutions, as well as Schauder's regularity theory for solutions in Hölder spaces. In this context, we also explain the continuity method that connects an equation that one wishes to study in a continuous manner with one that one understands already and deduces solvability of the former from solvability of the latter with the help of a priori estimates.

The final chapter develops the Moser iteration technique, which turned out to be fundamental in the theory of elliptic PDEs. With that technique one can extend many properties that are classically known for harmonic functions (Harnack inequality, local regularity, maximum principle) to solutions of a large class of general elliptic PDEs. The results of Moser will also allow us to prove the fundamental regularity theorem of de Giorgi and Nash for minimizers of variational problems.

At the end of each chapter, we briefly summarize the main results, occasionally suppressing the precise assumptions for the sake of saliency of the statements. I believe that this helps in guiding the reader through an area of mathematics that does not allow a unified structural approach, but rather derives its fascination from the multitude and diversity of approaches and methods, and consequently encounters the danger of getting lost in the technical details.

Some words about the logical dependence between the various chapters: Most chapters are composed in such a manner that only the first sections are necessary for studying subsequent chapters. The first—rather elementary—chapter, however, is basic for understanding almost all remaining chapters. Section 2.1 is useful, although not indispensable, for Chapter 3. Sections 4.1 and 4.2 are important for Chapters 6 and 7. Sections 8.1 to 8.4 are fundamental for Chapters 9 and 12, and Section 9.1 will be employed in Chapters 10 and 12. With those exceptions, the various chapters can be read independently. Thus, it is also possible to vary the order in which the chapters are studied. For example, it would make sense to read Chapter 8 directly after Chapter 1, in order to see the variational aspects of the Laplace equation (in particular, Section 8.1) and also the transformation formula for this equation with respect to changes of the independent variables. In this way one is naturally led to a larger class of elliptic equations. In any case, it is usually not very efficient to read a mathematical textbook linearly, and the reader should rather try first to grasp the central statements.

The present book can be utilized for a one-year course on PDEs, and if time does not allow all the material to be covered, one could omit certain sections and chapters, for example, Section 3.3 and the first part of Section 3.4 and Chapter 10. Of course, the lecturer may also decide to omit Chapter 12 if he or she wishes to keep the treatment at a more elementary level.

This book is based on a one-year course that I taught at the Ruhr University Bochum, with the support of Knut Smoczyk. Lutz Habermann carefully checked the manuscript and offered many valuable corrections and suggestions. The LaTeX work is due to Micaela Krieger and Antje Vandenberg.

The present book is a somewhat expanded translation of the original German version. I have also used this opportunity to correct some misprints in that version. I am grateful to Alexander Mielke, Andrej Nitsche, and Friedrich Tomi for pointing out that Lemma 4.2.3, and to C.G. Simader and Matthias Stark that the proof of Corollary 8.2.1 were incorrect in the German version.

Leipzig, Germany Jürgen Jost

Preface to the 2nd Edition

For this new edition, I have written a new chapter on reaction-diffusion equations and systems. Such equations or systems combine a linear elliptic or parabolic differential operator, of the type extensively studied in this book, with a non-linear reaction term. The result are phenomena that can be obtained by neither of the two processes – linear diffusion or non-linear reaction as in ordinary differential equations or systems – in isolation. The patterns resulting from this interplay of local non-linear self-interactions and global diffusion in space, such as travelling waves or Turing patterns, have been proposed as models for many biological and chemical structures and processes. Therefore, such reaction-diffusion systems are very popular in mathematical biology and other fields concerned with non-linear pattern formation. In mathematical terms, their success stems from the fact that, through a combination of the PDE techniques developed in this book and some dynamical systems methods, a penetrating and often rather complete mathematical analysis can be achieved. – This new chapter is inserted after Chapter 4 that deals with linear parabolic equations, since this is the area of PDEs that is basic for studying reaction-diffusion equations. While the new chapter thus finds its most natural place there, occasionally, we also need to invoke some results from subsequent chapters, in particular from §9.5 about eigenvalues of the Laplace operator. Still, we find it preferable to discuss reaction-diffusion equations and systems at this earlier place so that we can emphasize the parabolic diffusion phenomena. This chapter also provides us with the opportunity of a glimpse at systems of PDEs as opposed to single equations. That is, we study scalar functions each of which satisfies a PDE and which are coupled through non-linear interaction terms. Of course, the field of systems of PDEs is richer than this, and more difficult couplings are possible and important, but this seems to be the point to which we can reasonably get in an introductory textbook.

I have also rewritten §11.1 (§10.1 in the previous edition, but due to the insertion of the new chapter, subsequent chapter numberings are shifted in the present edition) on the Hölder regularity of solutions of the Poisson equation. The previous proof had a problem. While that problem could have been resolved, I preferred to write a new proof based on scaling relations that is perhaps more insightful than the previous one.

The new edition also contains numerous other additions, about Neumann boundary value problems, Poincaré inequalities, expansions,..., as well as some minor (mostly typographical) corrections. I thank some careful readers for relevant comments.

Leipzig, Aug.2006 Jürgen Jost

Contents

Introduction: What Are Partial Differential Equations?

As a first answer to the question, What are partial differential equations, we would like to give a definition:

Definition 1: *A partial differential equation (PDE) is an equation involving derivatives of an unknown function* $u\colon \Omega \to \mathbb{R}$, *where* Ω *is an open subset of* $\mathbb{R}^d$, $d \geq 2$ *(or, more generally, of a differentiable manifold of dimension* $d \geq 2$*).*

Often, one also considers systems of partial differential equations for vector-valued functions $u\colon \Omega \to \mathbb{R}^N$, or for mappings with values in a differentiable manifold.

The preceding definition, however, is misleading, since in the theory of PDEs one does not study arbitrary equations but concentrates instead on those equations that naturally occur in various applications (physics and other sciences, engineering, economics) or in other mathematical contexts.

Thus, as a second answer to the question posed in the title, we would like to describe some typical examples of PDEs. We shall need a little bit of notation: A partial derivative will be denoted by a subscript,

$$u_{x^i} := \frac{\partial u}{\partial x^i} \quad \text{for } i = 1, \dots, d.$$

In case $d = 2$, we write x, y in place of x^1, x^2. Otherwise, x is the vector $x = (x^1, \dots, x^d)$.

Examples: (1) The Laplace equation

$$\Delta u := \sum_{i=1}^{d} u_{x^i x^i} = 0 \quad (\Delta \text{ is called the Laplace operator}),$$

or, more generally, the Poisson equation

$$\Delta u = f \quad \text{for a given function} \quad f : \Omega \to \mathbb{R}.$$

For example, the real and imaginary parts u and v of a holomorphic function $u\colon \Omega \to \mathbb{C}$ ($\Omega \subset \mathbb{C}$ open) satisfy the Laplace equation. This easily follows from the Cauchy–Riemann equations:

$$\begin{aligned} u_x &= v_y, \\ u_y &= -v_x, \end{aligned} \quad \text{with} \quad z = x + iy$$

implies

$$u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}.$$

The Cauchy–Riemann equations themselves represent a system of PDEs. The Laplace equation also models many equilibrium states in physics, and the Poisson equation is important in electrostatics.

(2) The heat equation:
Here, one coordinate t is distinguished as the "time" coordinate, while the remaining coordinates $x^1, \dots, x^d$ represent spatial variables. We consider

$$u : \Omega \times \mathbb{R}^+ \to \mathbb{R}, \quad \Omega \text{ open in } \mathbb{R}^d, \quad \mathbb{R}^+ := \{t \in \mathbb{R} : t > 0\},$$

and pose the equation

$$u_t = \Delta u, \quad \text{where again } \Delta u := \sum_{i=1}^{d} u_{x^i x^i}.$$

The heat equation models heat and other diffusion processes.

(3) The wave equation:
With the same notation as in (2), here we have the equation

$$u_{tt} = \Delta u.$$

It models wave and oscillation phenomena.

(4) The Korteweg–de Vries equation

$$u_t - 6uu_x + u_{xxx} = 0$$

(notation as in (2), but with only one spatial coordinate x) models the propagation of waves in shallow waters.

(5) The Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = f,$$

or in higher dimensions

$$\det \left(u_{x^i x^j}\right)_{i,j=1,\dots,d} = f,$$

with a given function f, is used for finding surfaces (or hypersurfaces) with prescribed curvature.