Graduate Texts in Mathematics

Carlos A. Berenstein
Roger Gay

Complex Variables

An Introduction

复变导论

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To our families for their patience and encouragement

To Lars V. Ahlfors, with our admiration

Nadie puede escribir un libro. Para
Que un libro sea verdaderamente,
Se requieren la aurora y el poniente,
Siglos, armas y el mar que une y separa.

Jorge Luis Borges

Preface

Textbooks, even excellent ones, are a reflection of their times. Form and content of books depend on what the students know already, what they are expected to learn, how the subject matter is regarded in relation to other divisions of mathematics, and even how fashionable the subject matter is. It is thus not surprising that we no longer use such masterpieces as Hurwitz and Courant's *Funktionentheorie* or Jordan's *Cours d'Analyse* in our courses.

The last two decades have seen a significant change in the techniques used in the theory of functions of one complex variable. The important role played by the inhomogeneous Cauchy–Riemann equation in the current research has led to the reunification, at least in their spirit, of complex analysis in one and in several variables. We say reunification since we think that Weierstrass, Poincaré, and others (in contrast to many of our students) did not consider them to be entirely separate subjects. Indeed, not only complex analysis in several variables, but also number theory, harmonic analysis, and other branches of mathematics, both pure and applied, have required a reconsideration of analytic continuation, ordinary differential equations in the complex domain, asymptotic analysis, iteration of holomorphic functions, and many other subjects from the classic theory of functions of one complex variable. This ongoing reconsideration led us to think that a textbook incorporating some of these new perspectives and techniques had to be written. In particular, we felt that introducing ideas from homological algebra, algebraic topology, sheaf theory, and the theory of distributions, together with the systematic use of the Cauchy–Riemann $\bar{\partial}$-operator, were essential to a complete understanding of the properties and applications of the holomorphic functions of one variable.

The idea that function theory can be integrated into other branches of mathematics is not unknown to our students. It is our experience that undergraduates see many applications of complex analysis, such as the use of partial fractions, the Laplace transform, and the explicit computation of integrals and series which could not be done otherwise. Graduate students thus have a powerful motivation to understand the foundation of the theory of functions of one complex variable.

The present book evolved out of graduate courses given at the universities of Maryland and Bordeaux, where we have attempted to give the students a sense of the importance of new developments and the continuing vitality of the theory of functions. Because of the amount of material covered, we are presenting our work in two volumes.

We have tried to make this book self-contained and accessible to graduate students, while at the same time to reach quite far into the topics considered. For that reason we assume mainly knowledge that is found in the undergraduate curriculum, such as elementary linear algebra, calculus, and point set topology for the complex plane and the two-dimensional sphere S^2. Beyond this, we assume familiarity with metric spaces, the Hahn–Banach theorem, and the theory of integration as it can be found in many introductory texts of real analysis. Whenever we felt a subject was not universally known, we have given a short review of it.

Almost every section contains a large number of exercises of different levels of difficulty. Those that are not altogether elementary have been starred. Many starred exercises came from graduate qualifying examinations. Some exercises provide an insight into a subject that is explained in detail later in the text.

In the same vein, we have made each chapter, and sometimes each section, as independent as possible of the previous ones. If an argument was worth repeating, we did so. This is one of the reasons the formulas have not been numbered; when absolutely essential, they have been marked for ease of reference in their immediate neighborhood. There are some propositions and proofs that have also been starred, and the reader can safely skip them the first time around without loss of continuity. Finally, we have left for the second volume some subjects that require a somewhat better acquaintance with functional analysis.

Let us give a short overview of this volume. Some of the basic properties of holomorphic functions of one complex variable are really topological in nature. For instance, Cauchy's theorem and the theory of residues have a homotopy and a homology form. In the first chapter, we give a detailed description of differential forms (including a proof of the Stokes formula), homotopy theory, homology theory, and other parts of topology pertinent to the theory of functions in the complex plane. Later chapters introduce the reader to sheaf theory and its applications. We conclude Chapter 1 with the definition of holomorphic functions and with the properties of those functions that are immediate from the preceding topological considerations.

In the second chapter we study analytic properties of holomorphic functions, with emphasis on the notion of compact families. This permits an early proof of the Riemann mapping theorem, and we explore some of its consequences and extensions. The class S of normalized univalent functions is introduced as an example of a compact family. A one-semester course in complex analysis could very well start in this chapter and refer the student back to selected topics as necessary.

In the third chapter we consider the solvability of the inhomogeneous Cauchy–Riemann equation. As a corollary we obtain a simple exposition of ideal theory and corresponding interpolation theorems in the algebra of holomorphic functions. We also study the boundary values of holomorphic functions in the sense of distributions, showing that every distribution on $\mathbb{R}$ can be obtained as boundary value of a holomorphic function in $\mathbb{C}\backslash\mathbb{R}$. (An appendix to this chapter gives a short introduction to the concepts of distribution theory.) The Edge-of-the-Wedge theorem, an important generalization of the Schwarz reflection principle, is proven. These ideas lead directly to the theory of hyperfunctions to be considered in the second volume. We conclude this chapter with a totally new approach to the theory of residues.

In the fourth chapter we develop the theory of growth of subharmonic functions in such a way that Hadamard's infinite product expansion for entire functions of finite order is generalized to subharmonic functions. We give a proof due to Bell and Krantz of the fact that a biholomorphic mapping between smooth domains extends smoothly to the boundary. This is used to prove simply and rigorously properties of the Green function of a domain.

In order to develop fully the concept of analytic continuation, Chapter 5 has a short introduction to the theory of sheaves, covering spaces and Riemann surfaces. Among the applications of these ideas we give the index theorem for linear differential operators in the complex plane. This chapter also contains an introduction to the theory of Dirichlet series.

In the second volume the reader will find the application of the ideas and methods developed in the present volume to harmonic analysis, functional equations, and number theory. For instance, elliptic functions, mean-periodic functions, the corona theorem, the Bezout equation in spaces of entire functions, and the Leroy–Lindelöf theory of analytic continuation and its relation to functional equations and overconvergence of Dirichlet series.

This being a textbook, it is impossible to be entirely original, and we have benefited from the existence of many excellent monographs and even unpublished lecture notes, too many to give credit to all of them in every instance. The list of references contains their titles as well as those of a number of research articles relevant to the subjects we touched upon. In a few places we have also tried to steer the reader into further lines of study that were naturally related to the subject at hand, but that, due to the desire to keep this book within manageable limits, we were compelled to leave aside.

Almost everything that the reader will find in our book can be traced in one way or another to Ahlfors' *Complex Analysis*. When it appeared, it changed entirely the way the subject was taught. Although we do not aspire to such achievement, our sincere hope is that we have not let him down.

Finally, we would like to thank Virginia Vargas for the excellent typing and her infinite patience. A number of our friends and students, among them, F. Colonna, D. Pascuas, A. Sebbar, A. Vidras, and A. Yger, have gladly played the role of guinea pigs, reading different portions of the manuscript and offering excellent advice. Our heartfelt thanks to all of them.

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Contents

CHAPTER 1

Topology of the Complex Plane and Holomorphic Functions

§1. Some Linear Algebra and Differential Calculus

The complex plane $\mathbb{C}$ coincides with $\mathbb{R}^2$ by the usual identification of a complex number $z = x + iy$, $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, with the vector (x, y). As such it has two vector space structures, one as a two-dimensional vector space over $\mathbb{R}$ and the other as a one-dimensional vector space over $\mathbb{C}$. The relations between them lead to the classical Cauchy-Riemann equations.

Let $\mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$ be the space of all $\mathbb{R}$-linear maps from $\mathbb{C}$ into $\mathbb{R}$. These maps are also called (real) linear forms. It is clear that $\mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$ is an $\mathbb{R}$-vector space. Moreover, since $\{1, i\}$ forms an $\mathbb{R}$-basis for $\mathbb{C}$, the pair of linear forms

$$dx : h \mapsto \operatorname{Re} h \quad \text{and} \quad dy : h \mapsto \operatorname{Im} h$$

constitutes a basis for $\mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$.

Let $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$ denote the space of all $\mathbb{R}$-linear maps of $\mathbb{C}$ into itself. It is a vector space of dimension 4 over $\mathbb{R}$ and dimension 2 over $\mathbb{C}$. One way to see this is the following. The inclusion $\mathbb{R} \subseteq \mathbb{C}$ allows us to consider $\mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$ as an $\mathbb{R}$-linear subspace of $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$. We can decompose a form $L \in \mathscr{L}_{\mathbb{R}}(\mathbb{C})$ as $L = \operatorname{Re} L + i \operatorname{Im} L$. Hence, as real vector spaces

$$\mathscr{L}_{\mathbb{R}}(\mathbb{C}) = \mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R}) \oplus i\mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R}),$$

and we see immediately that $\dim_{\mathbb{R}} \mathscr{L}_{\mathbb{R}}(\mathbb{C}) = 4$. Moreover, any $\mathbb{R}$-basis of $\mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$ is a $\mathbb{C}$-basis of $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$ and, conversely, any $\mathbb{C}$-basis of $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$ consisting of real-valued mappings is an $\mathbb{R}$-basis for $\mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R})$. In particular, the pair $\{dx, dy\}$ is a $\mathbb{C}$-basis for $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$.

We shall consider now the complex subspace $\mathscr{L}_{\mathbb{C}}(\mathbb{C})$ of $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$ consisting of those linear forms which are $\mathbb{C}$-linear. Observe that a linear form $L = P\,dx + Q\,dy \in \mathscr{L}_{\mathbb{R}}(\mathbb{C})$ is $\mathbb{C}$-linear if and only if

$$L(ih) = iL(h)$$

for every $h \in \mathbb{C}$. Writing $h = h_1 + ih_2, h_1, h_2 \in \mathbb{R}$, we find $ih = -h_2 + ih_1$ and

$$L(ih) = -Ph_2 + Qh_1,$$

while

$$iL(h) = i(Ph_1 + Qh_2) = iPh_1 + iQh_2.$$

Therefore, $L \in \mathscr{L}_{\mathbb{C}}(\mathbb{C})$ if and only if $Q = iP$. Define the linear form

$$dz := dx + i\,dy,$$

$dz \in \mathscr{L}_{\mathbb{C}}(\mathbb{C})$ and $L = P\,dz$ whenever $Q = iP$. In particular, $\mathscr{L}_{\mathbb{C}}(\mathbb{C})$ has complex dimension 1 and real dimension 2.

Finally, let us denote by $\overline{\mathscr{L}_{\mathbb{C}}(\mathbb{C})}$ the subspace of $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$ of $\mathbb{C}$-antilinear transformations. That is, $L(\alpha h) = \bar{\alpha}L(h)$ for every $\alpha, h \in \mathbb{C}$. The involution $z \to \bar{z}$ can be extended from $\mathbb{C}$ to $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$, and it exchanges the subspaces $\mathscr{L}_{\mathbb{C}}(\mathbb{C})$ and $\overline{\mathscr{L}_{\mathbb{C}}(\mathbb{C})}$. It provides a direct sum decomposition (as real vector spaces):

$$\mathscr{L}_{\mathbb{R}}(\mathbb{C}) = \mathscr{L}_{\mathbb{C}}(\mathbb{C}) \oplus \overline{\mathscr{L}_{\mathbb{C}}(\mathbb{C})}.$$

The linear form $\overline{dz} = dx - i\,dy$ is in $\overline{\mathscr{L}_{\mathbb{C}}(\mathbb{C}}$ and it is usually denoted $d\bar{z}$. It is immediate to vertify that $\{dz, d\bar{z}\}$ is also a $\mathbb{C}$-basis of $\mathscr{L}_{\mathbb{R}}(\mathbb{C})$.

As an illustration of this, let us consider the formulas for the change of basis. When we write an element $L \in \mathscr{L}_{\mathbb{R}}(\mathbb{C})$ in terms of those two bases we have

$$L = P\,dx + Q\,dy = A\,dz + B\,d\bar{z},$$

where $P, Q, A, B \in \mathbb{C}$ are related by the equations

$$A = \tfrac{1}{2}(P - iQ), \qquad B = \tfrac{1}{2}(P + iQ)$$

$$P = A + B, \qquad Q = i(A - B).$$

The transformation L is $\mathbb{C}$-linear, i.e., $L \in \mathscr{L}_{\mathbb{C}}(\mathbb{C})$, if and only if $B = 0$. This is the familiar Cauchy-Riemann condition found earlier:

$$P = \frac{1}{i}Q.$$

When we identify $\mathbb{C}$ to $\mathbb{R}^2$, then L correpsonds to a 2×2 real matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$, related to the preceding representation by

$$P = a + ib, \qquad Q = c + id.$$

The Cauchy-Rieman condition takes the more familiar form of the pair of equations:

$$\begin{cases} a = d \\ b = -c. \end{cases}$$

Thus, the $\mathbb{C}$-linear transformation of multiplication by $P = a + ib \in \mathbb{C}$ has the matrix representation $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

It is also clear from these computations that $\mathscr{L}_{\mathbb{C}}(\mathbb{C}) \cap \mathscr{L}_{\mathbb{R}}(\mathbb{C}, \mathbb{R}) = \{0\}$, i.e., the only real-valued $\mathbb{C}$-linear transformation of $\mathbb{C}$ is the identically zero map.

We denote by $\mathscr{A}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$ the complex vector space of the alternating $\mathbb{R}$-bilinear mappings from $\mathbb{R}^2 \times \mathbb{R}^2$ into $\mathbb{C}$.

Recall that if $h = (h_1, h_2) \in \mathbb{R}^2$, $k = (k_1, k_2) \in \mathbb{R}^2$, and $B \in \mathscr{A}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$ then $h \to B(h,k)$ and $k \to B(h,k)$ are $\mathbb{R}$-linear and $B(h,k) = -B(k,h)$. An example of such a map is:

$$B(h,k) = \det\begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix} = h_1 k_2 - h_2 k_1.$$

We can generate other $\mathbb{R}$-bilinear maps by the following procedure: If $\phi, \theta \in \mathscr{L}_{\mathbb{R}}(\mathbb{C})$, then we define the ***wedge product*** (or ***exterior product***) $\phi \wedge \theta$, as the element in $\mathscr{A}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$ given by

$$(\phi \wedge \theta)(h,k) = \det\begin{pmatrix} \phi(h) & \theta(h) \\ \phi(k) & \theta(k) \end{pmatrix} = \phi(h)\theta(k) - \phi(k)\theta(h).$$

In this notation the previous example is simply $dx \wedge dy$.

Let us see that $\{dx \wedge dy\}$ is a $\mathbb{C}$-basis for $\mathscr{A}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$ and hence, $\dim_{\mathbb{C}} \mathscr{A}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C}) = 1$. It is evident that $dx \wedge dy \neq 0$. Moreover, elementary calculation shows that for any $B \in \mathscr{A}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$ we have

$$B = B(e_1, e_2)\, dx \wedge dy,$$

where $e_1 = (1,0)$ and $e_2 = (0,1)$.

One verifies that the mapping

$$\mathscr{L}_{\mathbb{R}}(\mathbb{C}) \times \mathscr{L}_{\mathbb{R}}(\mathbb{C}) \to \mathscr{A}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$$

$$(\phi, \theta) \mapsto \phi \wedge \theta$$

is also $\mathbb{R}$-bilinear and alternating. This proves the distributivity of the wedge product with respect to the sum and shows $\phi \wedge \phi = 0$ for every $\phi \in \mathscr{L}_{\mathbb{R}}(\mathbb{C})$. In particular,

$$dx \wedge dx = dy \wedge dy = dz \wedge dz = d\bar{z} \wedge d\bar{z} = 0,$$

$$dx \wedge dy = -dy \wedge dx,$$

and

$$dz \wedge d\bar{z} = -d\bar{z} \wedge dz = -2i\, dx \wedge dy,$$

which shows that $\{dz \wedge d\bar{z}\}$ is also a $\mathbb{C}$-basis for $\mathscr{A}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{C})$.

Let Ω be an open subset of $\mathbb{C}$ and E a normed space defined over $\mathbb{R}$. A mapping $f: \Omega \to E$ is said to be ***differentiable*** at $a \in \Omega$ if there is a linear transformation $L \in \mathscr{L}_{\mathbb{R}}(\mathbb{C}, E)$ (the space $\mathbb{R}$-linear transformations from $\mathbb{C}$ into E) such that for every $h \in \mathbb{C}$ of absolute value sufficiently small we have