Graduate Texts in Mathematics

R. W. Sharpe

Differential Geometry

Cartan's Generalization of Klein's Erlangen Program

微分几何

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Cartan's Generalization of Klein's Erlangen Program

Foreword by S.S. Chern

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Foreword

I am honored by Professor Sharpe's request to write a forward to his beautiful book.

In his preface he asks the innocent question, "Why is differential geometry the study of a connection on a principal bundle?" The answer is of course very simple; because Euclidean geometry studies a connection on a principal bundle, and all geometries are in a sense generalizations of Euclidean geometry.

In fact, let E^n be the Euclidean space of n dimensions. We call an orthonormal frame x, $e_1, \ldots, e_n$ ($n+1$ vectors), where x is the position vector and e_i have the scalar products

$$(e_i, e_j) = \delta_{ij}, \qquad 1 \leq i, j \leq n.$$

Then the space of all orthonormal frames is a principal fiber bundle with group $O(n)$ and base space E^n, the projection being defined by mapping x, $e_1, \ldots, e_n$ to x. The equations

$$de_i = \sum_{1 \leq j \leq n} \omega_{ij} e_j, \qquad 1 \leq i \leq n,$$

define the Maurer–Cartan forms ω_{ij}, with

$$\omega_{ij} + \omega_{ji} = 0, \qquad 1 \leq i, j \leq n.$$

They satisfy the Maurer–Cartan equations

$$d\omega_{ij} = \sum_{1 \leq k \leq n} \omega_{ik} \wedge \omega_{kj}, \qquad 1 \leq i, j \leq n.$$

This is Euclidean geometry by moving frames. The ω_{ij} define the parallelism or connection. The Maurer–Cartan equations say that the connection is flat. This formulation has a great generalization.

As in all disciplines, the development of differential geometry is tortuous. The basic notion is that of a manifold. This is a space whose coordinates are defined up to some transformation and have no intrinsic meaning. The notion is original, bold, and powerful. Naturally, it took some time for the concept to be absorbed and the technology to be developed. For example, the great mathematician Jacques Hadamard "felt insuperable difficulty ... in mastering more than a rather elementary and superficial knowledge of the theory of Lie groups," a notion based on that of a manifold [1]. Also, it took Einstein seven years to pass from his special relativity in 1908 to his general relativity in 1915. He explained the long delay in the following words: "Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning." [2]

On the technology side the breakthrough was achieved by the tensor analysis of Ricci calculus. The central theme was Riemannian geometry, which Riemann formulated in 1854. Its fundamental problem is the "form problem": To decide when two Riemannian metrics differ by a change in coordinates. This problem was solved by E. Christoffel and R. Lipschitz in 1870. Christoffel's solution introduces a covariant differentiation, which could be given an elegant geometrical setting through the parallelism of Levi-Civita. Tensor analysis is extremely effective and has dominated differential geometry for a century.

Another technical tool, which has not quite received the recognition it deserves, is the exterior differential calculus of Elie Cartan. This was introduced by Cartan in 1922, following the work of Frobenius and Darboux. All the exterior differential forms on a manifold form a ring. It depends only on the differentiable structure of the manifold and not on any additional structure such as a Riemannian metric or an affine connection. Topologically it leads to the de Rham theory. Less known is its effectiveness in treating local problems.

A fundamental question is the equivalence problem for G-structures: Given, on an n-dimensional manifold with coordinates u^i, a set of linear differential forms ω^i, a similar set ω^{*j} with coordinates u^{*j}, and a subgroup $G \subset Gl(n, \mathbf{R})$, determine the conditions under which there exist functions

$$u^{*j} = u^{*j}(u^1, \ldots, u^n), \qquad 1 \leq i, j \leq n,$$

such that after substitution the ω^{*j} differ from the ω^j by a transformation of G. The form problem in Riemannian geometry is the case $G = O(n)$.

The solution of the form problem by Cartan's method of equivalence leads automatically to the tensor analysis. Thus, the method of equivalence is more general. In the case $G = O(n)$, this leads to the Levi-Civita parallelism and the Riemannian geometry. In this way Euclidean geometry generalizes to Riemannian geometry. For a general G, the solution of the equivalence problem is not always easy (cf. the Preface), although it is proved that it can always be achieved in a finite number of steps. Philosophically nice problems have nice answers.

Klein geometry can be developed through the Maurer–Cartan equations. The generalization of the above discussion, from $O(n)$ to G, gives Cartan's generalized spaces, essentially a connection in a principal bundle.

A fundamental problem is the relation of the local geometry with the global properties of the spaces in question. Such a result is the so-called Chern–Weil theorem that the characteristic classes can be represented by differential forms constructed explicitly from the curvature. The simplest result is the Gauss–Bonnet formula.

I wish to take this occasion to mention some recent developments on Finsler geometry [3]. This is the geometry of a very simple integral and was discussed in problem 23 of Hilbert's Paris address in 1900. By a proper interpretation of the analytical results, Finsler geometry now assumes a very simple form showing it to be a family of geometries quite analogous to the Riemannian case.

Differential geometry offers an open vista of manifolds with structures, finite or infinite dimensional. There are also simple and difficult low-dimensional problems, of the garden variety. If one switches between the two, life is indeed very enjoyable.

It is a great mystery that the infinitesimal calculus is a source of such depth and beauty.

References

[1] J. Hadamard, *Psychology of Invention in the Mathematical Field*, Princeton University Press, Princeton, 1945, p. 115.

[2] A. Einstein, Autobiographical Notes, in *Albert Einstein: Philosopher Scientists*, p. 67, 2nd ed., 1949, in vol. 7 of the series The Library of Living Philosophers, Evanston, IL, edited by P.A. Schilpp.

[3] D. Bao and S.S. Chern, On a notable connection in Finsler geometry, *Houston J. Math.* **19** (1993), 135–180.

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Note on the Second Printing

This printing corrects many simple errata, but it does not deal with two issues, which will be treated in detail in a possible second edition of this book. For now I merely wish to indicate how they may be dealt with.

Part B of the structure theorem 2.8.3 is false as it stands. It becomes true if the hypothesis (iii) is replaced by the condition 'each leaf is compact'. I note that this result is never appealed to anywhere in the book.

In the proof of the characterization of Lie groups (Theorem 3.8.7) the maps $\mu: M \times M \to M$ and $\iota: M \to M$ on pages 131 and 133 are constructed by appealing to the fundamental theorem. But the target M is not as yet a Lie group so the fundamental theorem does not apply. What is needed is a version of the fundamental theorem for maps $f: N \to M$, where N is simply connected and M is a manifold satisfying the hypotheses of Theorem 3.8.7. I do not know how to prove such a result using the definition of a complete 1-form given on page 129. The remedy is to replace this definition by:

> **Definition 8.3.** Let V be a vector space and let ω be a V-valued 1-form on the smooth manifold M. Assume that $\omega_m: T_m M \to V$ is an isomorphism for every $m \in M$. We say that ω is *complete* if the vector field $(\omega^{-1}(f(t)), \partial_t)$ on $M \times \mathbf{R}$ is complete for every smooth function $f: \mathbf{R} \to V$.

The Maurer–Cartan form of a Lie group does indeed satisfy this condition, and with this stronger notion of completeness, Theorem 3.8.7 is true.

I would like to express my gratitude to Robert Solovay and Anthony Blaom for bringing these two issues (respectively) to my attention.

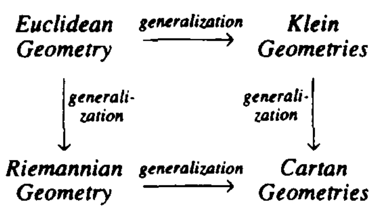
Richard Sharpe
Toronto, Canada
March 2, 2000

Preface

This book is a study of an aspect of Elie Cartan's contribution to the question "What is geometry?"

In the last century two great generalizations of Euclidean geometry appeared. The first was the discovery of the non-Euclidean geometries. These were organized into a coherent whole by Felix Klein, who recognized them as various examples of coset spaces G/H of Lie groups. In this book we refer to these latter as Klein geometries. The second generalization was Georg Riemann's discovery of what we now call Riemannian geometry. These two theories seemed largely incompatible with one other.[1]

In the early 1920s Elie Cartan, one of the pioneers of the theory of Lie groups, found that it was possible to obtain a common generalization of these theories, which he called *espaces généralizés* and we call Cartan geometries (see diagram).



[1]The only relationship was the "accident" that some of the non-Euclidean geometries could be regarded as special cases of Riemannian geometries.

Looking at this diagram vertically, we can say that just as a Riemannian geometry may be regarded, locally, as modeled on Euclidean space but made "lumpy" by the introduction of curvature, so a Cartan geometry may be regarded, locally, as modeled on one of the Klein geometries but made "lumpy" by the introduction of a curvature appropriate to the model in question. Looking at the same diagram horizontally, a Cartan geometry may be regarded as a non-Euclidean analog of Riemannian geometry.

Cartan actually gave the first example of a Cartan geometry more than a decade earlier, in the remarkable *tour de force* [E. Cartan, 1910]. In that paper he considered the case of a two-dimensional distribution on a five-dimensional manifold. He showed that such a distribution determined, and was determined by, a Cartan geometry modeled on the homogeneous space G_2/H, where H is a certain nine-dimensional subgroup of the fourteen-dimensional exceptional Lie group G_2. This process of associating a Cartan geometry to a raw geometric entity (the distribution) is an example of "solving the equivalence problem" for the entity in question. Although the solution of an equivalence problem is not always a Cartan geometry, in many important cases it is. When it is, the invariants of the geometry (curvature, etc.) are a priori invariants of the raw geometric entity. We recommend [R.B. Gardner, 1989] for an account of the method of equivalence.

To be a little more precise, a Cartan geometry on M consists of a pair (P, ω), where P is a principal bundle $H \to P \to M$ and ω, the Cartan connection, is a differential form on P. The bundle generalizes the bundle $H \to G \to G/H$ associated to the Klein setting, and the form ω generalizes the Maurer–Cartan form ω_G on the Lie group G. In fact, the *curvature* of the Cartan geometry, defined as $d\omega + \frac{1}{2}[\omega, \omega]$, is the complete local obstruction to P being a Lie group.

One reason for the power of Cartan's method comes from the fact that these new geometries maintain the same intimate relation with Lie groups that one sees in the case of homogeneous spaces. This means, for example, that constructions in the theory of homogeneous spaces often generalize in a simple manner to the general "curved" case of Cartan geometries. It also means that the differential forms that appear are always related to components of the Maurer–Cartan form of the Lie group, a context in which their significance remains clear.

In the particular case of a Riemannian manifold M, Cartan's point of view offered a new and profound vantage point that is largely responsible for the modern insistence on "doing differential geometry on the bundle P of orthonormal frames over M."

The history of the study of Cartan geometries is somewhat troubled. First is the difficulty Cartan faced in trying to express notions for which there was no truly suitable language.[2] Next is the widely noted difficulty in reading

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[2]This difficulty was resolved with the introduction of the notion of a principal bundle and of vector-valued forms on such a bundle.

Cartan.[3] In his paper [C. Ehresmann, 1950] Charles Ehresmann gave for the first time a rigorous global definition of a Cartan connection as a special case of a more general notion now called an Ehresmann connection (or more simply, a connection). For various reasons[4] the Ehresmann definition was taken as the definitive one, and Cartan's original notion went into a more or less total eclipse for a long time. The beautiful geometrical origin and insight connected with Cartan's view were, for many, simply lost. In short, although the Ehresmann definition gives us a good notion, it hides the real story about why it is so good. In this connection, the following quotation is interesting [S.S. Chern, 1979]:

> The physicist C.N. Yang wrote [C.N. Yang, 1977]: "That non-abelian gauge fields are conceptually identical to ideas in the beautiful theory of fibre bundles, developed by mathematicians without reference to the physical world, was a great marvel to me." In 1975 he mentioned to me: "This is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere."

Far from arising "out of nowhere," the simple and compelling geometric origin of a connection on a principal bundle is that it is a generalization of the Maurer–Cartan form. Moreover, a study of the Cartan connection itself can illuminate and unify many aspects of differential geometry.

Novelties

Aside from the fact that one cannot find a fully developed, modern exposition of Cartan connections elsewhere, what is new or different in this book?

New Treatment

This book is written at a level that can be understood by a first- or second-year graduate student. In particular, we include the relevant theory of manifolds, distributions and Lie groups. For us, a manifold is, by definition, a

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[3] To paraphrase Robert Bryant, "You read the introduction to a paper of Cartan and you understand nothing. Then you read the rest of the paper and still you understand nothing. Then you go back and read the introduction again and there begins to be the faint glimmer of something very interesting."

[4] At that stage it was easier to read Ehresmann than Cartan. There was also the attraction of a more general and global notion.

locally Euclidean, paracompact Hausdorff space. This is the same as a locally Euclidean Hausdorff space each of whose components has a countable basis.[5] In particular, Lie groups are defined to be manifolds in this sense. The result of Yamabe and Kuranishi ([H. Yamabe, 1950]) that a connected subgroup of a Lie group is a Lie subgroup implies that *any* subgroup of a Lie group is a Lie group in the present sense. The discussion of submanifolds given in Chapter 1 is broad enough to include these subgroups as submanifolds.

In our coverage of bundle theory, we emphasize the abstract principal bundles rather than bundles of frames.[6] Of course, these two views are really equivalent. In the case of the "first-order" geometries, the equivalence is quite simple. However, in the case of "higher-order" geometries, the choice of the higher-order frames usually seems to be decided on a rather ad hoc basis and can be complicated. Here the bundle approach gives a real advantage, and the right choice of frames becomes clear (if needed) once the bundle is understood. Another important advantage of working with the bundles themselves is that they give a common language, facilitating comparison between geometries and emphasizing the relation to the model space. In this sense, comparing Cartan geometries is like comparing Klein geometries.

Chapter 3 contains a complete and economical development of the Lie group—Lie algebra correspondence based on the *fundamental theorem of non-abelian calculus.* One of the novelties here is the characterization of a Lie group as a manifold equipped with a Lie algebra-valued form on it satisfying certain properties. This characterization prepares the reader for the generalization to Cartan geometries in Chapter 5.

Finally, in Appendix B we explain how one manifold may roll without slipping or twisting on another in Euclidean space. We also show how this notion yields a differential system that contains both the Levi–Civita connection and the Ehresmann connection on the normal bundle for a submanifold of Euclidean space.

New Results

Let us move on to some results we believe are new. In Chapter 4 we introduce the fundamental property of Klein geometries characterizing the kernel of such a geometry. This result is used in Chapter 5 in an essential way to show the equivalence of the base and bundle definitions of Cartan geometries in the effective case. In Chapter 5 we introduce and classify Cartan space forms. These geometries generalize the classical Riemannian

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[5]The usual definition requires a manifold to have a countable basis (cf., e.g., [Boothby, W. 1986, p. 6]).

[6]In much the same way, one might emphasize an abstract Lie group rather than a matrix group realizing it.

space forms.[7] One important ingredient of this classification is the property (apparently new) of a Cartan geometry called "geometric orientability." Another is the notion of "model mutation." Finally, in Chapter 7 we give a classification of the submanifolds of a Möbius geometry. This classification is more general than that of [A. Fialkow, 1944] in that ours allows the presence of umbilic points.

Prerequisites and Conventions

This book assumes very few prerequisites. The reader needs to be familiar with some basic ideas of group theory, including the notion of a group acting on a set. Results from the calculus of several variables, point set topology, and the theory of covering spaces are used in various places, and the long, exact sequence of homotopy theory is used once (at the end of Chapter 5). Aside from this, most of the material is developed *ab initio*. However, the reader is invited to shoulder some of the burden of the work in that essential use is made of a few of the exercises. These exercises are denoted by an asterisk to the right of the exercise number.

The numbering follows a single sequence throughout the book, with all items (definitions, theorems, figures, etc.) in a single stream. Thus 4.3.2 refers to Chapter 4, Section 3, item 2. For references to items occurring in the same chapter, we omit the chapter number, so that in Chapter 4, 4.3.2 becomes 3.2.

We use the following dictionary of symbols to denote the ends of various items:

symbol	*end of*
❀	definition
❑	exercise
■	proof
◆	example

Although it will often be convenient for us to write column vectors as row vectors, the reader should remember that all vectors are in fact column vectors.

[7] In fact, this notion is general enough to immediately allow a description of general symmetric spaces.

Limitations

The reader will find no mention here of some basic topics in differential geometry, such as Stokes' theorem, characteristic classes, and complex geometries. Also, our approach to Lie theory is "elementary" in that we do not discuss or use the classification theory of Lie groups, with its attendant study of roots, weights, and representations.

Originally, we had wished to include more than the three examples of Cartan geometries studied here; but in the end, the pressures of time, space, and energy limited this impulse. The three geometries we do study are not developed in complete analogy to each other. For example, the discussion of immersed curves in a Möbius geometry in terms of the normal forms given in Chapter 7 does of course have a Riemannian analog, but that is not studied in this book. And one may study subgeometries of projective geometries just as one studies subgeometries of Riemannian and Möbius geometries, but we do not do so here. We have also resisted the impulse to make a "dictionary" translating among the various versions of Cartan's view, Ehresmann's view,[8] and the view expressed in [L.P. Eisenhart, 1964]. In the end, however, for those who are interested in it, it should be abundantly clear how Cartan's view does illuminate the others.

Some Personal Remarks

An author often writes a book in order to sort out his or her own understanding of the subject. This is the circumstance in the present case. When I was an undergraduate, differential geometry appeared to me to be a study of curvatures of curves and surfaces in $\mathbf{R}^3$. As a graduate student I learned that it is the study of a connection on a principal bundle. I wondered what had become of the curves and surfaces, and I studied topology instead.

The reawakening of my interest in this subject began in 1987 when Tom Willmore very kindly wrote me a note thanking me for a preprint and mentioning his great interest in what is known as the Willmore conjecture (cf. 7.6). This led me once again to look at principal bundles and connections. In particular, I wondered whether there was an intrinsically defined Ehresmann connection on a surface in S^3 that was invariant under the group of Möbius transformations of S^3. It turns out there is no such connection. However, after calculating normal forms for surfaces in the Möbius sphere S^3 (cf. [G. Cairns, R. Sharpe, and L. Webb, 1994]), it became clear to me that there must be some other kind of invariantly defined structure inherited on the surface from its embedding in S^3. (In Chapter 7 it is shown

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[8]See, however, the discussion in Appendix A dealing with the relationship between Cartan and Ehresmann connections.

that a Cartan connection is defined in this situation, and, in fact, Cartan also knew this [E. Cartan, 1923].)

During this time it began to seem strange to me that Ehresmann connections play such a prominent role in modern differential geometry. In some cases, such as the Levi-Civita connection, the connection is determined by the geometry. In many cases, however, one makes use of an arbitrary connection that one proves to exist by a general technique. This is the appropriate point of view for the construction of the characteristic classes of Chern and Pontryagin. There one may use any connection, since the aim is to obtain topological invariants for which the particular choice of connection does not matter. But these considerations seem to be at their base topological rather than differential geometric. My innocent question, left over from my undergraduate days, was "Why is differential geometry the study of a connection on a principal bundle?" And I began, rather impertinently, to ask this question at every opportunity, usually picking on some unsuspecting differential geometer who did not know me very well.

During one of these sessions, Min Oo remarked that Elie Cartan had considered connections with values in a Lie algebra larger than that of the fiber.[9] Later I read, and translated,[10] Cartan's book [E. Cartan, 1935]. I browsed through Cartan's collected works and through those of his successors and interpreters. It became clear to me that Cartan had a subtle and really wonderful idea, which gives a fully satisfying explanation for the modern, and approximately true, notion that differential geometry is the study of an Ehresmann connection on a principal bundle. There seems to be no treatment of these things in the standard texts on differential geometry. In the few books where the Cartan connections are mentioned at all (e.g., [J. Dieudonné 1974], [W.A. Poor, 1981], and [M. Spivak, 1979]), they make only a brief appearance, perhaps in the exercises or toward the end of the book, and one is left with the impression that the notion is only a quaint curiosity left over from bygone days. Six years ago I began to scribble some notes about these things and to talk about them; after a number of months had passed, I realized I was writing a book on the subject.

* * *

I would like to thank everyone who has had an influence on this book. In addition to those mentioned above, I am grateful to Bernard Kamte, Joe Repka, Qunfeng Yang, and my wife, Mary, for their comments on portions of the manuscript. My thanks go to all the staff at Springer-Verlag,

[9]See [E. Ruh, 1993] for a brief recent overview of Cartan connections and some of their applications.

[10]A copy of my translation, which is only a rough draft, can be found in the Mathematics Library at the University of Toronto.