系统科学与数学

JOURNAL OF SYSTEMS SCIENCE AND MATHEMATICAL SCIENCES

卷(Vol.) 1　　期(No.) 1

1981

《系统科学与数学》编辑委员会编辑

科 学 出 版 社 出 版

178 x 1.

系统科学与数学

（季 刊）

一九八一年 第1卷 第1期

编 辑 系统科学与数学编辑委员会

编辑部在北京中关村中国科学院系统科学研究所内

出 版 科 学 出 版 社

北京朝阳门内大街137号

印刷装订 中 国 科 学 院 印 刷 厂

总发行处 新 华 书 店 北 京 发 行 所

订 购 处 全 国 各 地 新 华 书 店

北京市期刊登记证第611号

统一书号：13031.1680 本社书号：2303·13-1

定价：0.82 元

系统科学与数学

第 1 卷　第 1 期

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J. Sys. Sci. & Math. Scis.
1 (1) (1981), 1-8

多目标规划的理想点法

应玫茜

1. 引言

假定X是n维欧氏空间E_n的子集，在X上有m个目标函数$F_i(x)$ $(i=1,\cdots,m)$，对每个$F_i(x)$事先规定了一个理想值F_i^*（例如$F_i^*=\min\limits_{x\in X}F_i(x)$）. 一般而言，不一定存在$x^0\in X$，使$F_i^*=F_i(x^0)$ $(i=1,\cdots,m)$，但我们可以求得$\bar{x}\in X$，使向量$(F_1(\bar{x}),F_2(\bar{x}),\cdots,F_m(\bar{x}))$和向量$(F_1^*,\cdots,F_m^*)$尽量地接近. 这时可以通过求解

$$\min_{x\in X}\sum_{i=1}^{m}\lambda_i[F_i(x)-F_i^*]^2$$

（其中λ_i为根据$F_i(x)$的重要程度而决定的权系数，$\sum\limits_{i=1}^{m}\lambda_i=1$，$\lambda_i>0$，$i=1,\cdots,m$. 如不能比较$F_i(x)$之间的重要程度，则取$\lambda_i=1$，$i=1,\cdots,m$），得到最优解$\bar{x}$（解这种类型的方法一般简称之为理想点法[1]），易证$\bar{x}$是多目标问题的有效解（即不存在$x^*\in X$，使$F_i(x^*)\leqslant F_i(\bar{x})$，$i=1,\cdots,m$，且在不等式中至少有一个取严格的不等号）. 由此可见，求与多目标规划理想点距离最近的点问题可归结为求带约束集合X的最小平方和问题.

在文[2]中，我们曾给出两个不同的带约束最小平方和的解法. 在解法中有两个问题值得进一步考虑，一是迭代的初始点是否必须取自约束集？当约束条件较复杂时，要找出一个真正的可行解来往往是很困难的；二是迭代中要求单变量问题的最优解往往是很困难的，工作量很大，是否可用某种近似解来代替最优解，使工作量大大减少且不影响方法的收敛性？本文给出一个解法解决了上述两个问题，且在与[2]类似的假设下，证明了它的收敛性.

2. 解法

考虑下述最小平方和问题

$$(P)\begin{cases}\min\limits_{x\in R}\sum\limits_{i=1}^{m}f_i^2(x),\\ R=\{x\in E_n\,|\,g_j(x)\geqslant 0,\ j=1,\cdots,l\}.\end{cases}$$

假设

(i) $f_i(x)(i=1,\cdots,m)$，$g_j(x)(j=1,\cdots,l)$有一阶连续偏导数.

(ii) R有界，内点集$\mathring{R}=\{x\,|\,g_j(x)>0,\ j=1,\cdots,l\}$非空.

(iii) $-g_j(x)(j=1,\cdots,l)$为凸函数.

1981年1月21月收到.

为方便起见，引入下述符号

$$f(x)=(f_1(x),\cdots,f_m(x))^T,$$

$$g(x)=(g_1(x),\cdots,g_l(x))^T,$$

$$\nabla f(x)=\begin{pmatrix}\frac{\partial f_1(x)}{\partial x_1}\cdots\frac{\partial f_1(x)}{\partial x_n}\\ \vdots \qquad \vdots\\ \frac{\partial f_m(x)}{\partial x_1}\cdots\frac{\partial f_m(x)}{\partial x_n}\end{pmatrix},$$

$$\nabla g(x)=\begin{pmatrix}\frac{\partial g_1(x)}{\partial x_1}\cdots\frac{\partial g_1(x)}{\partial x_n}\\ \vdots \qquad \vdots\\ \frac{\partial g_l(x)}{\partial x_1}\cdots\frac{\partial g_l(x)}{\partial x_n}\end{pmatrix},$$

$$\|f(x)\|^2=\sum_{i=1}^m f_i^2(x),\quad \|\cdot\| \text{为欧氏模}.$$

引入罚函数

$$\phi_M(x)=\frac{1}{2}\|f(x)\|^2-M\sum_{j=1}^l \min(g_j(x),0),$$

其中 $M\geqslant 1$ 为参数.

设 C 为 $n\times n$ 对称正定矩阵，任意取定 x，考虑二次正定规划

$$P(x,C)\begin{cases}\min\limits_z \frac{1}{2}z^T((\nabla f(x))^T\nabla f(x)+C)z+(f(x))^T\nabla f(x)z,\\ \nabla g(x)z+g(x)\geqq 0.\end{cases}$$

由假设 (ii)(iii)，易证 $P(x,C)$ 的约束集非空. 由 [3] 知 $P(x,C)$ 的最优解一定存在（且唯一）.

在第 k 次迭代时，取定矩阵 C_k 满足

$$a\|z\|^2\leqslant z^TC_kz\leqslant b\|z\|^2,$$

其中 $b>a>0$，a 与 b 为任意取定的与 k 无关的常数（例如取 $C_k=\alpha_k I$，$\alpha_k>0$，I 为单位矩阵）.

在迭代前先取定下列参数：

① 取一适当大的罚参数 $M\geqslant 1$；

② 取适当小的罚函数单变量问题最优解的搜索区间长 $\beta>0$；

③ 取定罚函数最小值允许误差数 $\varepsilon_k>0$，要求 $\sum\limits_{k=0}^{\infty}\varepsilon_k<+\infty$.

迭代步骤

1° 任取初始点 $x^0\in E_n$；令 $k:=0$.

2° 设已知 x^k. 求 $P(x^k,C_k)$ 的最优解 z^k. 若 $z^k=0$，则迭代停止，x^k 即为所求；若

$z^k \neq 0$, 则求 $\lambda_k > 0$ 使

$$\phi_M(x^k + \lambda_k z^k) \leqslant \min_{0 \leqslant \lambda \leqslant \beta} \phi_M(x^k + \lambda z^k) + \varepsilon_k.$$

令 $x^{k+1} = x^k + \lambda_k z^k$, $k := k+1$, 转入 2° 的开始.

3. 收敛性

首先,为要说明步骤 2° 中 λ_k 的存在性，必须证明当 $z^k \neq 0$ 时，$\phi_M(x)$ 在 x^k 处沿方向 z^k 的方向导数

$$D_z\phi_M(x^k) = \lim_{\lambda \to 0+} \frac{\phi_M(x^k + \lambda z^k) - \phi_M(x^k)}{\lambda} < 0,$$

这时,当 $\lambda > 0$ 充分小时,就有

$$\phi_M(x^k + \lambda z^k) < \phi_M(x^k),$$

因此可以在 $\lambda = 0$ 适当小的邻域内找出 λ_k. 为此证明下述引理.

引理 1. 在假设 (i) 之下，设 $(z, u) \in E_n \times E_l$ 满足 $P(x, C)$ 的 Kuhn-Tucker 条件(简称 (z, u) 为 KT 点). 若 $z \neq 0$, $u_j < M(j = 1, \cdots, l)$, 则

$$D_z\phi_M(x) < 0.$$

证. 由方向导数的性质可知

$$\begin{aligned} D_z\phi_M(x) &= \frac{1}{2} D_z\|f(x)\|^2 - M \sum_{j=1}^{l} D_z \min(g_j(x), 0) \\ &= (f(x))^T \nabla f(x) z - M \sum_{j:\, g_j(x) < 0} \nabla g_j(x) z \\ &\quad - M \sum_{j:\, g_j(x) = 0} \min(\nabla g_j(x) z, 0). \end{aligned} \tag{1}$$

作 $P(x, C)$ 的 Lagrange 函数

$$\begin{aligned} L(z, u) = \frac{1}{2} z^T((\nabla f(x))^T \nabla f(x) + C) z + (f(x))^T \nabla f(x) z \\ - u^T(\nabla g(x) z + g(x)). \end{aligned}$$

已知 (z, u) 满足 Kuhn-Tucker 条件

$$0 = L_z = z^T((\nabla f(x))^T \nabla f(x) + C) + (f(x))^T \nabla f(x) - u^T \nabla g(x), \tag{2}$$

$$0 \geqq L_u = -g(x) - \nabla g(x) z, \tag{3}$$

$$0 = u^T L_u = -u^T(g(x) + \nabla g(x) z), \; u \geqq 0. \tag{4}$$

由 (2), (4) 有

$$\begin{aligned} (f(x))^T \nabla f(x) z &= -z^T((\nabla f(x))^T \nabla f(x) + C) z + u^T \nabla g(x) z \\ &\leqslant -u^T g(x) \\ &\leqslant -\sum_{j:\, g_j(x) < 0} u_j g_j(x). \end{aligned} \tag{5}$$

由 (3), 当 $g_j(x) \leqslant 0$ 时,有

$$\nabla g_j(x) z \geqslant -g_j(x) \geqslant 0. \tag{6}$$

故由 (1),(5),(6), 当 $u_j \leqslant M$ $(j = 1, \cdots, l)$ 时,

$$D_z\phi_M(x) = (f(x))^T\nabla f(x)z - M\sum_{j:g_j(x)<0}\nabla g_j(x)z$$
$$\leqslant \sum_{j:g_j(x)<0}(M-u_j)g_j(x) < 0.$$

证毕.

引理 2[4]. 设 y^i 为二次规划

$$\begin{cases}\min\left(\frac{1}{2}y^TD_iy + p_i^Ty\right) & (i=1,2),\\ A^iy - b^i \geqslant 0 & (i=1,2)\end{cases}$$

的最优解. 若存在 y^* 使 $A^1y^* - b^1 > 0$, 则必存在 $\rho > 0$, $\varepsilon^* > 0$, 当

$$\varepsilon = \max\{\|D_1 - D_2\|, \|A^1 - A^2\|, \|p^1 - p^2\|, \|b^1 - b^2\|\} \leqslant \varepsilon^*$$

时,有

$$\|y^1 - y^2\| \leqslant \rho\varepsilon.$$

定理 1. 在假设 (i) 之下, 若 $P(x^k, C_k)$ 的乘子 $u_j^k \leqslant M(j=1,\cdots,l)$ 对一切 k 成立. 则或者经有限次迭代后得到问题(P)的 KT 点, 或者在点列 $\{x^k\}$ 的极限点中若有 $\bar{x}$ 使集合 $\{z\,|\,\nabla g(\bar{x})z + g(\bar{x}) > 0\}$ 非空,则此 $\bar{x}$ 必为问题 (P) 的 KT 点的 x 分量.

证. 设 z^k 为 $P(x^k, C_k)$ 的最优解. 由 Kuhn-Tucker 定理,知存在乘子 u^k 使 (z^k, u^k) 为 $P(x^k, C_k)$ 的 KT 点,即 $x = x^k$, $z = z^k$, $u = u^k$, $C = C_k$ 满足 (2),(3),(4). 由此,易见若 $z^k = 0$, 则 (x^k, u^k) 满足问题 (P) 的 KT 条件,即 (x^k, u^k) 为问题 (P) 的 KT 点.

若 $z^k \neq 0$ 对一切 k 成立. 由引理 1, x^k 对一切 k 都存在. 设 $\bar{x}$ 为 $\{x^k\}$ 的极限点使集合 $\{z\,|\,\nabla g(\bar{x}) + \nabla g(\bar{x})z > 0\}$ 非空. 由 C_k 的取法,可不妨设

$$\lim_{k\to\infty} x^k = \bar{x}, \quad \lim_{k\to\infty} C_k = \bar{C}, \quad \bar{C} \text{ 正定}.$$

设 $P(\bar{x}, \bar{C})$ 的 KT 点为 $(\bar{z}, \hat{u})$. 则 $x = \bar{x}$, $z = \bar{z}$, $u = \hat{u}$, $C = \bar{C}$ 满足 (2), (3), (4). 若 $\bar{z} = 0$, 则 $(\bar{x}, \hat{u})$ 即为问题 (P) 的 KT 点.

若 $\bar{z} \neq 0$. 由引理 2 以及 $f_i(x)$, $\nabla f_i(x)$, $g_j(x)$, $\nabla g_j(x)$ 的连续性,易见

$$\lim_{k\to\infty} z^k = \bar{z}.$$

因 $0 \leqslant u_j^k \leqslant M$ $(j=1,\cdots,l)$ 对一切 k 成立,故点列 $\{u^k\}$ 必有极限点, 设为 $\bar{u}$. 由 (2)(3)(4), 知 $(\bar{z}, \bar{u})$ 为 $P(\bar{x}, \bar{C})$ 的 KT 点.

设 $\bar{\lambda} > 0$ 为 $\min\limits_{0\leqslant\lambda\leqslant\beta}\phi_M(\bar{x} + \lambda\bar{z})$ 的最优解. 由引理 1, 知 $\phi_M(\bar{x} + \bar{\lambda}\bar{z}) < \phi_M(\bar{x})$. 令

$$\gamma = \phi_M(\bar{x}) - \phi_M(\bar{x} + \bar{\lambda}\bar{z}), \quad \gamma > 0.$$

因 $\lim\limits_{k\to\infty}(x^k + \bar{\lambda}z^k) = \bar{x} + \bar{\lambda}\bar{z}$, 故对充分大的 k, 有

$$\phi_M(x^k + \bar{\lambda}z^k) < \phi_M(\bar{x} + \bar{\lambda}\bar{z}) + \frac{\gamma}{2} = \phi_M(\bar{x}) - \frac{\gamma}{2}.$$

由 x^k 的取法,当 k 充分大时有

$$\sum_{j=k}^{\infty}\varepsilon_j < \frac{\gamma}{2}, \quad \phi_M(x^{k+1}) < \phi_M(x^k) + \varepsilon_k,$$

由此得

$$\phi_M(\bar{x}) < \phi_M(x^{k+1}) + \sum_{j=k+1}^{\infty} \varepsilon_j \leqslant \phi_M(x^k + \bar{\lambda} z^k) + \varepsilon_k + \sum_{j=k+1}^{\infty} \varepsilon_j$$

$$< \phi_M(x^k + \bar{\lambda} z^k) + \frac{\gamma}{2} < \phi_M(\bar{x}) - \frac{\gamma}{2} + \frac{\gamma}{2} = \phi_M(\bar{x}),$$

由此得出矛盾．故 $\bar{z} = 0$．证毕．

定理 2．在假设 (i),(ii),(iii) 之下，设 $n \times n$ 对称矩阵 C 满足 $0 < a\|z\|^2 \leqslant z^T C z \leqslant b\|z\|^2$ (对一切 $z \neq 0$)，设 X 为有界闭集．则存在 $M > 0$，使对任何 $x \in X$，$P(x, C)$ 的乘子 u 满足

$$0 \leqslant u_j \leqslant M \quad (j = 1, \cdots, l).$$

证．由假设 (ii)，知存在 x^*，使 $g(x^*) > 0$．由假设 (iii)，对任意取定的 $x \in X$，有

$$0 < g(x^*) \leqq g(x) + \nabla g(x)(x^* - x), \tag{7}$$

即 $z^0 = x^* - x$ 为 $P(x, C)$ 的可行解．

设 (z, u) 为 $P(x, C)$ 的 KT 点．由鞍点定理以及 KT 条件 (4)，$u^T(\nabla g(x) z + g(x)) = 0$，有

$$\frac{1}{2} z^T((\nabla f(x))^T \nabla f(x) + C) z + (f(x))^T \nabla f(x) z \leqslant \frac{1}{2} z^{0T}((\nabla f(x))^T \nabla f(x) + C) z^0 + (f(x))^T \nabla f(x) z^0 - u^T(\nabla g(x) z^0 + g(x)). \tag{8}$$

故由 (7)(8) 以及 $u \geqq 0$，有

$$u^T g(x^*) \leqslant u^T(\nabla g(x) z^0 + g(x))$$
$$\leqslant \frac{1}{2} z^{0T}((\nabla f(x))^T \nabla f(x) + C) z^0 + (f(x))^T \nabla f(x) z^0 - \frac{1}{2} z^T((\nabla f(x))^T \nabla f(x) + C) z - (f(x))^T \nabla f(x) z. \tag{9}$$

我们知道 $P(x, C)$ 的对偶规划为

$$\begin{cases} \max\limits_{z, v} \left[\frac{1}{2} z^T((\nabla f(x))^T \nabla f(x) + C) z + (f(x))^T \nabla f(x) z - v^T(\nabla g(x) z + g(x))\right], \\ ((\nabla f(x))^T \nabla f(x) + C) z + (\nabla f(x))^T f(x) - \nabla g(x)^T v = 0, \\ v \geqq 0, \end{cases}$$

消去变量 z，易见上述对偶规划等价于

$$\begin{cases} \max\limits_{v} \left[-\frac{1}{2}((f(x))^T \nabla f(x) - v^T \nabla g(x))((\nabla f(x))^T \nabla f(x) + C)^{-1}((\nabla f(x))^T f(x) - (\nabla g(x))^T v) - v^T g(x)\right], \\ v \geqq 0. \end{cases}$$

由弱对偶定理，取 $v = 0$，有

$$\frac{1}{2} z^T((\nabla f(x))^T \nabla f(x) + C) z + (f(x))^T \nabla f(x) z$$
$$\geqslant -\frac{1}{2}(f(x))^T \nabla f(x)((\nabla f(x))^T \nabla f(x) + C)^{-1}(\nabla f(x))^T f(x),$$

故由 (9)，有

$$u^T g(x^*) \leqslant \frac{1}{2} z^{0T}((\nabla f(x))^T \nabla f(x) + C) z^0 + (f(x))^T \nabla f(x) z^0$$
$$+ \frac{1}{2} (f(x))^T \nabla f(x)((\nabla f(x))^T \nabla f(x) + C)^{-1} (\nabla f(x))^T f(x). \tag{10}$$

令

$$\delta_1 = \min_{1 \leqslant j \leqslant l} g_j(x^*), \quad \delta_2 = \max_{x \in X} \|x - x^*\|,$$

$$\delta_3 = \max \begin{cases} \max\limits_{x \in X} \|(\nabla f(x))^T \nabla f(x) + C\|, \\ \max\limits_{x \in X} \|(f(x))^T \nabla f(x)\|, \\ \max\limits_{x \in X} \|((\nabla f(x))^T \nabla f(x) + C)^{-1}\|. \end{cases}$$

易见 $\|z^0\| \leqslant \delta_2$, $\delta_1 u_j \leqslant u^T g(x^*)$ $(j = 1, \cdots, l)$. 故由 (10),

$$\delta_1 u_j \leqslant \frac{1}{2} \delta_2^2 \delta_3 + \delta_2 \delta_3 + \frac{1}{2} \delta_3^3 \quad (j = 1, \cdots, l).$$

令 $M = \frac{1}{2} \delta_1^{-1} \delta_2^2 \delta_3 + \delta_1^{-1} \delta_2 \delta_3 + \frac{1}{2} \delta_1^{-1} \delta_3^3$, 即得欲证. 证毕.

定理 3. 在假设 (i), (ii), (iii) 之下, 对任何初始点 x^0, 存在 $M_0 \geqslant 1$, 当 $M \geqslant M_0$ 时, 或者经有限次迭代后得到 x^k 为问题 (P) 的 KT 点的 x 分量, 或者点列 $\{x^k\}$ 的任何极限点 $\bar{x}$ 为问题 (P) 的 KT 点的 x 分量.

证. 由假设 (ii), (iii), 对任何 $x \in E_n$, 有

$$\nabla g(x)(x^* - x) + g(x) \geqq g(x^*) > 0,$$

故由定理 1, 只须证明 $P(x^k, C_k)$ 的乘子 u^k 有界. 由定理 2, 若能证明 $\{x^k\}$ 属于一个有界闭集, 则可得 $\{u^k\}$ 有界的结论, 这时取

$$M_0 = \max\left\{1, \frac{1}{2} \delta_1^{-1} \delta_2^2 \delta_3 + \delta_1^{-1} \delta_2 \delta_3 + \frac{1}{2} \delta_1^{-1} \delta_3^3\right\},$$

当 $M \geqslant M_0$ 时定理成立.

由 λ_k 的取法, 知

$$\phi_M(x^{k+1}) \leqslant \phi_M(x^k) + \varepsilon_k \leqslant \phi_M(x^0) + \sum_{i=0}^{k} \varepsilon_i,$$

即当 $M \geqslant M_0$ 时, 对一切 k 有

$$-\sum_{j=1}^{l} \min(g_j(x^{k+1}), 0) \leqslant \frac{1}{2M} \|f(x^0)\|^2 - \frac{1}{2M} \|f(x^{k+1})\|^2$$
$$- \sum_{j=1}^{l} \min(g_j(x^0), 0) + \frac{1}{M} \sum_{i=0}^{k} \varepsilon_i$$
$$\leqslant \frac{1}{2} \|f(x^0)\|^2 - \sum_{j=1}^{l} \min(g_j(x^0), 0) + \sum_{i=0}^{\infty} \varepsilon_i = \rho$$

$(\rho > 0$, 与 k 无关). 显见

$$-\sum_{j=1}^{l} \min(g_j(x^0), 0) \leqslant \rho.$$

作集合

$$X=\left\{x \,\middle|\, -\sum_{j=1}^{l}\min\left(g_j(x),\,0\right)\leqslant\rho\right\}.$$

上面已经证明 $\{x^k\}\subseteq X$. 令

$$h(x)=\sum_{j=1}^{l}\min\left(g_j(x),\,0\right),$$

由 $g(x)$ 的连续性，知 $h(x)$ 为连续函数，故 X 为闭集（实际上对任何 $\rho>0$ 都为闭集）. 又由 $-g_j(x)$ 的凸性，知 $-h(x)$ 为凸函数，故 X 又为凸集.

由于

$$X_0=\{x\,|\,h(x)\geqslant 0\}=\{x\,|\,g(x)\geqq 0\}=R,$$

由假设 (ii), X_0 为有界闭集. 显见 $X_0\subseteq X$.

最后欲证 X 有界. 若 X 无界，即存在 $y^k\in X$, $\lim\limits_{k\to\infty}\|y^k\|=+\infty$. 任取 $\tilde{x}\in X_0$, 由于 X 为凸集，故对任意 $\lambda\geqslant 0$, 有

$$\tilde{x}+\frac{\lambda}{\|y^k\|}(y^k-\tilde{x})\in X\left(当\ 0\leqslant\frac{\lambda}{\|y^k\|}\leqslant 1\right).$$

由于点列 $\{(y^k-\tilde{x})/\|y^k\|\}$ 有界，不妨设

$$\lim_{k\to\infty}\frac{y^k-\tilde{x}}{\|y^k\|}=y,\quad \|y\|=1.$$

由于 X 为闭集，故

$$\tilde{x}+\lambda y\in X\quad（一切\ \lambda\geqslant 0）.$$

又因 X_0 有界，$X_0\subseteq X$，故必存在 $\bar{\lambda}>0$ 使

$$\hat{x}=\tilde{x}+\bar{\lambda}y\bar{\in}X_0,$$

即 $-\rho\leqslant h(\hat{x})<0$. 又因 $-h(x)$ 为凸函数，由

$$\hat{x}=\lambda\left[\tilde{x}+\frac{1}{\lambda}(\hat{x}-\tilde{x})\right]+(1-\lambda)\tilde{x},\quad 0<\lambda<1$$

得

$$h(\hat{x})\geqslant\lambda h\left(\tilde{x}+\frac{1}{\lambda}(\hat{x}-\tilde{x})\right)+(1-\lambda)h(\tilde{x}).$$

由此对充分小的 $\lambda>0$, 有

$$-\rho\leqslant h\left(\tilde{x}+\frac{1}{\lambda}(\hat{x}-\tilde{x})\right)\leqslant\frac{1}{\lambda}\left[h(\hat{x})-(1-\lambda)h(\tilde{x})\right]$$

$$\leqslant\frac{1}{\lambda}h(\hat{x})<-\rho,$$

由此得到矛盾. 故 X 为有界闭集. 证毕.

定理 4. 在假设 (i),(ii),(iii) 之下，若 $\|f(x)\|^2$ 为伪凸函数，则或者经有限次迭代后得到 x^k 为问题 (P) 的最优解，或者点列 $\{x^k\}$ 的任何极限点为问题 (P) 的最优解.

证. 这是推广后的 Kuhn-Tucker 定理[5]以及定理 3 的直接推论. 证毕.

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AN IDEAL POINT METHOD FOR MULTIOBJECTIVE PROGRAMMING

YING MEIQIAN

ABSTRACT

Let $F_1(x), \cdots, F_m(x)$ be m objectives, where x is an n-dimensional decision variable vector. Let $X = \{x \in E_n \mid g_j(x) \geqslant 0, j = 1, \cdots, l\}$ be a constraint set. For each $F_i(x)$, we have an ideal value F_i^*. We know that the optimal solution of the problem

$$(\mathrm{P}) \qquad \min_{x \in X} \sum_{i=1}^{m} \lambda_i [F_i(x) - F_i^*]^2$$

is an efficient solution of multiobjective programming $\min\limits_{x \in X} (F_1(x), \cdots, F_m(x))$, where $\lambda_1, \cdots, \lambda_m$ are weighted coefficients. In this paper, we give a convergent algorithm to solve the problem (P).

系统科学与数学
J. Sys. Sci. & Math. Scis.
1 (1) (1981), 9–33

Ba SPACES AND SOME ESTIMATES OF LAPLACE OPERATOR

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INTRODUCTION

This paper is divided into two parts. In the first part, a new class of function spaces, called Ba, is introduced, and some of its properties are investigated.

This class of spaces contains the well-known classical Orlicz space, Orlicz-Sobolev space and so on, with many properties similar to those of Orlicz space, but they possess more contents. As the investigation is based on the classical Lebesgue class L_p, our methods are more natural and the results more concrete. We have applied this kind of spaces to the error estimates of difference methods and to the calculus of variations with strong nonlinearity. We only concentrate on some basic properties of the spaces Ba, such as separability, compactness, representation of linear functional, weak convergence, etc. for future applications.

In the second part we have obtained some *a* priori estimates of linear PDE in the new spaces Ba. We establish these estimates for the Laplace operator in some classical domains such as the whole space E_n and the half space E^+_{n+1}. We have also got some results for the 1/4 space (In the case of plane it corresponds to the first quadrant). The boundaries of these domains are characteristically not smooth, i.e. with angles. Recently this kind of domains attract many authors, for they have important applications in finite element method. So we shall discuss them in a particular case, as they may be interesting in the theory of finite element method. We find a certain discrete phenomenon in the a priori estimates of the angular domain. The estimates do not hold for some differentiable function spaces of particular fractional order.

To make them hold, we must add some auxiliary conditions about the known data. B. A. Солонников also found such phenomenon for the mixed problem of a heat equation. But for the Laplace operator this phenomenon is important because it may influence the convergence and error estimate of the finite element method.

I. SPACES Ba, aB AND $\widehat{Ba}$

§ 1. *Spaces Ba and aB*

We introduce a new class of function spaces.

Let $B = \{B_1, \cdots, B_m, \cdots\}$ be a sequence of linear normed function spaces, $a = \{a_1, \cdots, a_m, \cdots\}$ be a sequence of non-negative numbers. $\phi(z) = \sum a_m z^m$ is an integral

Received January 23, 1981.

function. For $f \in \bigcap_m B_m$, we form the power sevies of

$$I(f,\alpha)=\sum_{m=1}^{\infty} a_m \alpha^m \|f\|_{B_m}^m. \tag{I.1}$$

If $I(f,\alpha)$ has a non-zero radius of convergence, we say $f \in BL(\phi)$, or simply denote $BL(\phi) \equiv Ba \equiv (B_m, a_m)$. We also denote $I(f,1)$ by $I(f)$.

The norm in Ba is defined by

$$\|f\|_{Ba}=\inf_{I(f,\alpha)\leqslant 1}\left\{\frac{1}{\alpha}\right\}, \tag{I.2}$$

imply denoted by $\|f\|$.

It is easy to prove that $\|f\|$ satisies the three axioms of norms.

Theorem 1. *If B_m are* Banach *function spaces and any fundamental sequence of B_m contains a* p. p. *convergent subsequence, then Ba is also a* Banach *function space.*

Proof. We need only prove the completeness. Suppose $\{f_n\}$ is a fundamental sequence n Ba, i.e. for any $\varepsilon>0$, there exists $N(\varepsilon)$, such that for $n, l > N(\varepsilon)$, we have

$$\|f_n - f_l\| < \varepsilon.$$

Evidently $\{f_n\}$ is also a fundamental sequence of each B_m. So there exists a p. p. Convergent subsequence $\{f_{n_k}\}$, which is not only a fundamental sequence of Ba, but also that of each B_m.

Now we have

$$\sum_{m=1}^{N} a_m \frac{\|f_{n_{k_1}} - f_{n_{k_2}}\|_{B_m}^m}{\|f_{n_{k_1}} - f_{n_{k_2}}\|^m} \leqslant 1.$$

For any $\varepsilon>0$, there exists K, such that for $k_1, k_2 > K$, we have

$$\|f_{n_{k_1}} - f_{n_{k_2}}\| < \varepsilon < 1.$$

Therefore

$$\sum_{m=1}^{N} a_m \|f_{n_{k_1}} - f_{n_{k_2}}\|_{B_m}^m \leqslant \|f_{n_{k_1}} - f_{n_{k_2}}\| < \varepsilon.$$

Putting $k_2 \to \infty$ on the left side we have

$$\sum_{m=1}^{N} a_m \|f_{n_{k_1}} - f\|_{B_m}^m \leqslant \varepsilon.$$

Hence

$$\sum_{m=1}^{\infty} a_m \|f_{n_{k_1}} - f\|_{B_m}^m \leqslant \varepsilon,$$

and

$$f_{n_{k_1}} - f \in Ba.$$

And we get $f \in Ba$.

Now since, if $k_2, k_3 \to \infty$,

$$\big| \|f_{n_{k_1}} - f_{n_{k_2}}\| - \|f_{n_{k_1}} - f_{n_{k_3}}\| \big| \leqslant \|f_{n_{k_2}} - f_{n_{k_3}}\| \to 0,$$

we come to

$$\|f_{n_{k_1}} - f_{n_{k_2}}\| \to \lambda_{k_1} \leqslant \varepsilon, \quad \text{if } k_2 \to \infty.$$

For fixed N,

$$\sum_{m=1}^{N} a_m \frac{\|f_{n_{k_1}} - f_{n_{k_2}}\|_{B_m}^m}{\|f_{n_{k_1}} - f_{n_{k_2}}\|^m} \leqslant 1.$$

Putting $k_2 \to \infty$, we get

$$\sum_{m=1}^{N} a_m \frac{\|f_{n_{k_1}} - f\|_{B_m}^m}{\lambda_{k_1}^m} \leqslant 1.$$

Again putting $N \to \infty$, we obtain

$$\|f_{n_{k_1}} - f\| \leqslant \lambda_{k_1} \leqslant \varepsilon.$$

This proves $f_{n_{k_1}} \Longrightarrow f$.

Because $\{f_{n_k}\}$ is a subesequence of the fundamental sequence $\{f_n\}$,

$$f_n \Longrightarrow f.$$

The theorem is proved.

Evidently if $B_m = L_m$, then $Ba \equiv$ Orlicz space $L(\phi)$, which we denote by La. If $B_m = W^l_m$, then $Ba \equiv W^l L(\phi)$, which we denote by $W^l a$.

If $I(f, \alpha)$ is an integral function of α, then we say $f \in BE(\phi)$. We denote $BE(\phi) \equiv aB \equiv (a_m, B_m)$, where the space aB is a subspace of Ba.

We say the space B_m possesses the absolutely continuous norm if

(i) For any measurable subset E, $f \in B_m$. Then $\chi(E) f \in B_m$, where $\chi(E)$ is the characteristic function of E.

(ii) For any given $\varepsilon > 0$, there exists δ, such that for any measurable subset E, mes $E < \delta$. Then

$$\|\chi(E) f\|_{B_m} < \varepsilon.$$

Define

$$\|f\|_{B_m(E)} = \|\chi(E) f\|_{B_m}.$$

Now suppose all B'_ms possess absolutely continuous norms, and moreover if $|f_1| \leqslant |f_2|$, we have $\|f_1\|_{B_m} \leqslant \|f_2\|_{B_m}$.

Then we are led to

Theorem 2. *If $f \in Ba$, the necessary and sufficient condition for $f \in aB$ is the existence of a sequence of bounded functions $f_n \in aB$, such that*

$$\|f - f_n\| \to 0.$$

Proof. i) If $f \in aB$, putting

$$f_n = \begin{cases} f, & |f| \leqslant n, \\ 0, & \text{otherwise}, \end{cases}$$

$$E_n = \{x \mid f(x) > n\},$$

we have

$$I(\lambda|f - f_n|) = \sum_{m=1}^{N} a_m \|\lambda(f - f_n)\|_{B_m}^m + \sum_{m=N+1}^{\infty} a_m \|\lambda f\|_{B_m(E_n)}^m = I_1 + I_2.$$

Taking $\lambda > \frac{2}{\varepsilon}$, for sufficiently large N, we have $I_2 \leqslant \frac{1}{2}$. As I_1 possesses only finite terms, from the absolute continuities of the norms of B_m's and the measurability of f, we get $I_1 \leqslant \frac{1}{2}$ for $n > n_0$, where n_0 is sufficiently large, and therefore

$$I(\lambda|f - f_n|) \leqslant 1.$$

But because

$$I\left(\frac{\lambda|f - f_n|}{I(\lambda|f - f_n|) + 1}\right) \leqslant \frac{I(\lambda|f - f_n|)}{I(\lambda|f - f_n|) + 1} \leqslant 1,$$

so

$$\lambda\|f - f_n\| \leqslant I(\lambda|f - f_n|) + 1 \leqslant 2.$$

And we induce

$$\|f - f_n\| \leqslant \frac{2}{\lambda} < \varepsilon.$$

So f can be approximately approached by bounded f_n.

ii) If f is the Ba limit of bounded functions $f_n \in aB$, then $f \in aB$.

In fact for any fixed λ and sufficiently large n, we have

$$\lambda\|f - f_n\| < \varepsilon \leqslant \frac{1}{2} \leqslant 1.$$

Then

$$1 \geqslant I\left(\frac{\lambda|f - f_n|}{\lambda\|f - f_n\|}\right) = \sum a_m \frac{\lambda^m\|f - f_n\|_{B_m}^m}{\lambda^m\|f - f_n\|^m}$$

$$\geqslant \sum a_m \frac{\lambda^m\|f - f_n\|_{B_m}^m}{\lambda\|f - f_n\|^m},$$

$$I(\lambda|f - f_n|) \leqslant \lambda\|f - f_n\| < \varepsilon.$$

$$I(\lambda f) = I\left(\frac{2\lambda f_n + 2\lambda(f - f_n)}{2}\right) \leqslant \frac{1}{2}\left[I(2\lambda|f_n|) + I(2\lambda|f - f_n|)\right]$$

$$\leqslant \frac{1}{2} I(2\lambda|f_n|) + \varepsilon,$$

So $I(\lambda f) < \infty$, and we have $f \in aB$.

§ 2. *The Case of* $B_m = L_{p_m}$

In the following we shall consider the special case B_m = Lebesgue class L_{p_m}, where $p_m > 1$, and discuss properties of the space Ba.

1. Separabitity of the spaces aB

Suppose $u(x)$ is a bounded measurable function, $|u(x)| \leqslant a$. Then by the theorem of Luzin, we can find continuous functions $u_n(x)$ such that $|u_n(x)| \leqslant a$ and that $u(x) - u_n(x)$ may be different from zero only on the sets $G_n \subset G$, where $\operatorname{mes} G_n < \frac{1}{n}$.

Then we get

$$\begin{aligned} I(u-u_n, \alpha) &= \sum a_m(\|u-u_n\|_{p_m})^m \alpha^m = \sum a_m(\|u-u_n\|_{p_m(G)})^m \alpha^m \\ &= \sum a_m \left(\int_{G_n} |u-u_n|^{p_m} dx\right)^{m/p_m} \alpha^m \leqslant \sum a_m (\operatorname{mes} G_n)^{m/p_m} (2a)^m \alpha^m \\ &< \sum a_m \left(\frac{1}{n}\right)^{m/p_m} (2a\alpha)^m \\ &= \sum_{m=1}^{N} a_m \left(\frac{1}{n}\right)^{m/p_m} (2a\alpha)^m + \sum_{m=N+1}^{\infty} a_m (2a\alpha)^m = I_1 + I_2. \end{aligned}$$

For any given $\varepsilon > 0$ there exists N_1. When $N > N_1$, we have $I_2 < \frac{\varepsilon}{2}$. Then fixing N and making n sufficiently large, we get $I_1 < \frac{\varepsilon}{2}$. So

$$I(u-u_n, \alpha) \to 0, \ \forall \alpha, \text{ when } n \to \infty,$$

i.e. for any $\varepsilon > 0$, there exists N, such that when $n > N$,

$$\|u-u_n\| < \varepsilon, \quad \left(\text{Taking } \alpha = \frac{1}{\varepsilon}\right).$$

Hence we have

$$\|u-u_n\| \to 0.$$

In other words, continuous functions are dense everywhere in aB. Because every continuous function $u(x)$ can be uniformly approximated by polynomials with rational coefficients, these polynomials are dense in aB. So aB is separable.

2. Inequality

Denote by $\Pi(aB, r)$ the functions $u(x) \in aB$ such that

$$d(u, aB) = \inf_{w \in aB} \|u-w\| < r.$$

Then we have

$$\Pi(aB, 1) \subset \widehat{Ba} \subset \overline{\Pi}(aB, 1),$$

where

$$\widehat{Ba} = \{f \mid f \in Ba, I(f) < \infty\}.$$

Proof. Suppose $u(x) \in Ba$ and $\|u - u_0\| < 1$ for $u_0 \in aB$. Then there exists $\alpha > 0$ such that $\|u - u_0\| < 1 - \alpha$. Because the space aB is linear,

$$\frac{1}{\alpha} u_0(x) \in aB \subset Ba.$$

Therefore

$$\frac{\|u - u_0\|}{1 - \alpha} < 1, \quad \frac{u - u_0}{1 - \alpha} \in \widehat{Ba}.$$

From the convexity of $\widehat{Ba}$, we see

$$u(x) = (1 - \alpha) \frac{u(x) - u_0(x)}{1 - \alpha} + \frac{\alpha}{\alpha} u_0(x) \in \widehat{Ba}.$$

The left half of the inequality is proved.

Now take $u \in \widehat{Ba}$, then there exists a bounded function $u_\varepsilon(x) \in aB$ such that

$$I(u - u_\varepsilon) < \varepsilon,$$

$$I\left(\frac{u - u_\varepsilon}{1 + \varepsilon}\right) < \frac{1}{1 + \varepsilon} I(u - u_\varepsilon) < \frac{\varepsilon}{1 + \varepsilon} < 1,$$

$$\|u - u_\varepsilon\| < 1 + \varepsilon.$$

Hence we have

$$d(u, aB) = \inf_{w \in aB} \|u - w\| \leqslant \|u - u_\varepsilon\| \leqslant 1 + \varepsilon.$$

But ε is arbitrary, so

$$d(u, aB) \leqslant 1.$$

The right half of the inequatity is proved.

3. The absolute continuity of norms of the space aB

Suppose $u(x) \in aB$. For any $\varepsilon > 0$, there exists a bounded function $u_1(x)$ such that

$$\|u - u_1\| < \frac{\varepsilon}{2}.$$

Then we have

$$\|u\chi(G_1)\| \leqslant \|u - u_1\| + a\|\chi(G_1)\|,$$

where $\chi(G_1)$ denotes the characteristic function of $G_1 \subset G$. Without loss of generality we can suppose mes $G_1 < 1$,

$$I(\chi(G_1), \alpha) = \sum a_m \,(\text{mes } G_1)^{m/p_m} \alpha^m$$

$$\leqslant \sum_{m=1}^{N} a_m (\text{mes } G_1)^{m/p_m} \alpha^m + \sum_{m=N+1}^{\infty} a_m \alpha^m.$$

From this, for any fixed $\alpha > 0$, given small ε, there exists δ, when mes$G_1 < \delta$, $I(\chi(G_1), \alpha) \leqslant \varepsilon$. Then for any $\varepsilon > 0$, there exists δ, such that when mes$G_1 < \delta$, we have

$$a\|\chi(G_1)\| < \frac{\varepsilon}{2},$$