Some Topics in Paratopological and Semitopological Groups

仿拓扑群和半拓扑群的若干专题

李丕余　谢利红　牟　磊　薛昌涛　著

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Preface

Topology began with the investigation of certain questions in geometry and it was an area of mathematics concerned with properties that are preserved under continuous deformations of objects, such as deformations that involve stretching, but no tearing or gluing. In general, we could consider these concepts in topology such as continuity and convergence. For algebra, we all know that an algebraic structure on a set always considers the rules of operations and relations on itself. Among all these operators, groups and semigroups are very popular in mathematics. Together with geometry, analysis, topology, combinatorics and number theory, algebra is one of the main branches of pure mathematics. With the development of modern mathematics, sometimes mathematicians are asked to consider the topology and algebra structure together such as when studying functional analysis, dynamical systems, representation theory and others. In the 20th century many topologists and algebraists have contributed to topological algebra. Some outstanding mathematicians were involved such as Dieudonné J, Pontryagin L S, Weil A and Weyl H. An important topic of topological algebra is the interplay between topological and algebraic structures on the same set; in particular, the way in which otherwise inequivalent topological properties are forced to become equivalent by the presence of a compatible algebraic structure, and vice versa. For example, if the underlying topological space of a topological group satisfies the T_0 separation axiom, then it is automatically a Hausdorff space (and even completely regular). Clearly, the answer to the question how the relationship between topological properties depends on the underlying algebraic structure should strongly depend on the way the algebraic structure is related to the topology. The weaker the restrictions on the connection between topology and algebraic structure are, the larger is the class of objects included in the theory. Because of this, even when our main interest is in topological groups, it is natural to consider more general objects such as paratopological groups and semitopological groups which are more generalized than topological groups. Examples we encounter in such a larger class of objects help us to understand the beautiful properties of topological groups better.

Our main concern here will be paratopological and semitopological groups. The thorough *topological* study in paratopological and semitopological groups began about twenty years ago ([24–26] [80]). Arhangel'skiĭ A V , Tkachenko M published their book *Topological Groups and Related Structures* in 2008. This book summed up many results established in topology algebra and posed a lot of open problems which pointed out the direction in studying paratopological and semitopological groups. In comparison with topolgical groups, the theory of paratopological and semitopological groups is quite different. Many concepts in the theory of paratopological and semitopological groups have no analogues in topological groups at all. For example, the closure of a subgroup might not be again a subgroup and this is false even for first countable Abelian paratopological groups ([17], Example 1.4.17). The most famous example of a paratopological group, the Sorgenfrey line, shows that first countable (hereditarily) normal paratopological groups need not be metrizable. Now the theory of paratopological and semitopological groups is a dynamically developing branch of Mathematics with its own technique.

In 2009, Shen Rongxin and Lin Shou published the paper *A note on generalized metrizable properties in topological groups* ([90]) which was the first paper in this field in China. In the past few years, topologists Arhangel'skiĭ A V, Comfort W W, van Mill J and Tkachenko M came to China to have academic communications in succession. Lin Fucai's book *Topology algebra and generalized metric spaces*([55]) almost covered all area of topology algebra, especially the excellent work in China before 2013. This book collects the most recently work of the authors in paratopological and semitopological groups.

We expect readers to know the basic facts from general topology and topology algebra. The standard reference books are [17] [35] [41] [65].

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Li Piyu, Xie Lihong, Mou Lei and Xue Changtao

September 2014

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Chapter 1

Introduction

Given an object with an algebraic structure, say, a group, and a topology on it, one can require distinct types of relation between them. An important topic of topological algebra is the interplay between topological and algebraic structures on the same set. In this chapter, we will introduce some definitions in topology and topological algebra and some related simple properties will be given.

We use $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ to denote the natural numbers, the integers, the rational numbers, and the real numbers,respectively. The symbols ω and ω_1 denote the first infinite ordinal and uncountable ordinal, respectively.

Below $c(X)$, $d(X)$, $w(X)$, $nw(X)$, $l(X)$, and $k(X)$ denote the cellularity, density, weight, network weight, Lindelöf degree, and compact-covering number of a space X defined, respectively, as follows:

Cellularity: $c(X) = \sup\{|\mathcal{U}| : \mathcal{U} \text{ is a disjoint family of open subsets of } X\} + \omega$.

Density: $d(X) = \min\{|S| : S \subset X \text{ and } \overline{S} = X\} + \omega$.

Weight: $w(X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a base for } X\} + \omega$.

Network weight: $nw(X) = \min\{|\mathcal{U}| : \mathcal{U} \text{ is a network for } X\} + \omega$.

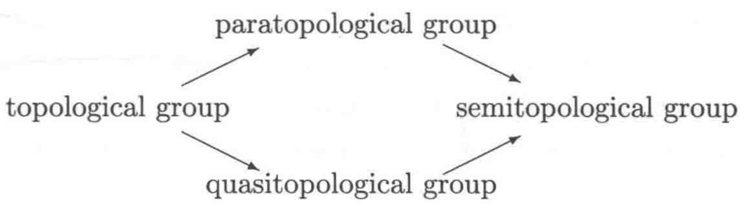
Lindelöf degree: $l(X) = \min\{\lambda \in \text{Card} : \text{for every open cover } \mathcal{V} \text{ of } X$ there is a subfamily $\mathcal{U} \subset \mathcal{V}$ such that $|\mathcal{U}| \leqslant \lambda$ and $\bigcup\mathcal{U} = X\} + \omega$.

Compact-covering number: $k(X) = \min\{\lambda \in \text{Card} : \mathcal{U}$ is a family of compact subsets of X such that $|\mathcal{U}| \leqslant \lambda$ and $\bigcup\mathcal{U} = X\} + \omega$.

A collection of nonempty open sets $\mathcal{U}$ of X is called a π-base if for every nonempty open set O, there exists an $U \in \mathcal{U}$ such that $U \subseteq O$. A *π-base* of a space at a point x of X is a family γ of non-empty open subsets of X such that every open neighborhood of x contains at least one element of γ. Put $\pi_\chi(x, X) = \min\{|\gamma| : \gamma \text{ is a } \pi\text{-base at } x\} + \omega$. Then the π-character of X is $\pi_\chi(X) = \sup\{\pi_\chi(x, X) : x \in X\}$.

A *semitopological group* is a group with a topology such that the multiplication in the group is separately continuous. If G is a semitopological group and the inverse operation of G is continuous, then G is called a *quasitopological group.* A *paratopological group* is a group with a topology such that the multiplication is jointly continuous. If G is a paratopological group and the inverse operation of G is continuous, then G is called a *topological group.*

It is clear from the definitions that



For a semitopological group G with identity e we will consider the following cardinal functions:

Character: $\chi(G) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a neighborhood base at } e \text{ of } G\} + \omega$.

Pseudocharacter: $\psi(G) = \min\{|\mathcal{U}| : \mathcal{U}$ is a family of open subsets of G such that $\bigcap \mathcal{U} = \{e\}\} + \omega$.

Example 1.1.1. Let τ be the topology on $\mathbb{R}$ with the base $\mathcal{B}$ consisting of the sets $[a, b) = \{x \in \mathbb{R} : a \leqslant x < b\}$, where $a, b \in \mathbb{R}$ and $a < b$. With the topology and the natural addition, $\mathbb{R}$ is a paratopological group. However, $(\mathbb{R}, \tau)$ is not a topological group since the inverse operation $x \to -x$ is not continuous. This paratopological group is called Sorgenfrey line.

Example 1.1.2. Let $G = (\mathbb{R}^2, +)$ be the group with the usual addition. Endow G with the topology which has a base $\{U(\langle s,t\rangle, \varepsilon, \delta) : (s,t) \in G, \varepsilon > 0, \delta > 0\}$, where $U(\langle s,t\rangle, \varepsilon, \delta) = \{\langle s,t\rangle\} \bigcup \{\langle s', t'\rangle : 0 < |s - s'| < \varepsilon, |(t' - t)/(s' - s)| < \delta\}$ ([41], Example 9.10). It is easy to see that G is a regular quasitopological group. However, G is not a topological group since $\langle s_1, t_1\rangle + \langle s_2, t_2\rangle \to \langle s_1 + s_2, t_1 + t_2\rangle$

is not continuous.

Let G be a group. For a fixed element $a \in G$, the mapping $x \to ax$ and $x \to xa$ of G onto itself are called the *left* and *right translations* of a on G, and are denoted by λ_a and ϱ_a, respectively.

Proposition 1.1.3. Let G be a semitopological group and g be any element of G. Then:

a) the right translation ϱ_g of G by g is a homeomorphism of the space G onto itself;

b) for any base $\mathcal{B}_e$ of the space G at e, the family $\mathcal{B}_g = \{Ug : U \in \mathcal{B}_e\}$ is a base of G at g.

Proof. Clearly, a) implies b). To prove a), we just observe that, in a semitopological group, every right translation ϱ_g is a continuous bijection. Since $\varrho_g \circ \varrho^{-1}$ is the identity mapping, it follows that the inverse of ϱ_g is also continuous, that is, ϱ_g is a homeomorphism of G onto itself. □

With the same method, we can prove that the corresponding result for the left translation λ_a of G.

Recall that a topological space X is said to be homogeneous if for each $x \in X$ and each $y \in X$, there exists a homeomorphism f of the space X onto itself such that $f(x) = y$. From Proposition 1 we easily obtain the next result:

Corollary 1.1.4. Every semitopological group is a homogeneous space.

Proof. Let G be a semitopological group. Take any elements x and y in G, and put $z = x^{-1}y$. Then $\varrho_z(x) = xz = xx^{-1}y = y$. Since, by Proposition 1, ϱ_z is a homeomorphism, the space G is homogeneous. □

It is well known that every T_0 topological group is Tychonoff. The influence of separation axioms on the topology of a paratopological group is a considerably more subtle issue than it appears in the case of topological groups. It is an old question whether every regular paratopological group is Tychonoff. The examples below show that neither of the implications

$$T_0 \Rightarrow T_1 \Rightarrow T_2 \Rightarrow \textit{regular}$$

is valid in paratopological groups, independently of their algebraic structure.

Example 1.1.5. $T_0 \nRightarrow T_1$. Let $\mathbb{R}$ be the additive group of real numbers endowed with the topology $\tau_0 = \{[x, +\infty) : x \in \mathbb{R}\}$. Then (R, τ_0) is a first-countable T_0 paratopological group which fails to be a T_1 space.

Example 1.1.6. $T_1 \nRightarrow T_2$. Let $\mathbb{Z}$ be the additive group of integers. Consider the topology $\tau_1 = \{x + [y, +\infty) : x, y \in \mathbb{Z}\}$ on $\mathbb{Z}$. Then $(\mathbb{Z}, \tau_1)$ is a T_1 paratopological group in which every two basic open sets have a nonempty intersection. Hence the space $(\mathbb{Z}, \tau_1)$ is not Hausdorff.

Example 1.1.7. $T_2 \nRightarrow$ *regular*. Let $\mathbb{R}^2$ be the additive group and $\mathcal{B}_0 = \{\langle 0, 0\rangle\} \cup \{\langle x, y\rangle \in \mathbb{R}^2 : |x| < 1/n, 0 < y < 1/n\}$ a local base at the zero element $\langle 0, 0\rangle$. Consider the topology τ_2 generated by the base $\mathcal{B} = \{\langle x, y\rangle + B : x, y \in \mathbb{R}$ and $B \in \mathcal{B}_0\}$. It is clear that $(\mathbb{R}^2, \tau_2)$ is a first-countable Hausdorff paratopological group which is not regular.

Next, we will introduce some concepts in the theory of generalized metric spaces.

Moore space theory has greatly influenced the theory of generalized metric spaces, so we will often refer to Moore spaces, and relate them to other classes of generalized metric spaces. Let $\{\mathcal{U}_n : n \in \omega\}$ be a sequence of open subsets of a space X. Recall that, for every $x \in X$ and n, $st(x, \mathcal{U}_n) = \bigcup\{U \in \mathcal{U}_n : x \in U\}$.

A sequence of open covers $\{\mathcal{U}_n : n \in \omega\}$ of a space X is called a *development*, if for every $x \in X$, the sequence $\{st(x, \mathcal{U}_n) : n \in \omega\}$ is a base at x. A space with a development is called a *developable space*. A *Moore space* is a regular developable space.

A space X is *quasi-developable* if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of families of open subsets of X such that for each $x \in X$, $\{st(x, \mathcal{U}_n) : n \in \omega\}$ is a base at x. This definition is the same as that of developable spaces, except that the $(\mathcal{U}_n)$'s do not have to cover the space. It was shown that a space X is developable if and only if X is perfect (=closed sets are G_δ-sets) and quasi-developable ([41], Theorem 8.6).

The G_δ-diagonal property is a simple property which appears as a factor in many theorems characterizing metrizable or developable spaces. A space X has a G_δ-diagonal if the set $\Delta = \{(x, x) \in X \times X : x \in X\}$ is a G_δ-subset in $X \times X$. The following characterization of spaces having a G_δ-diagonal is very useful for relating them to other classes of generalized metric spaces. It shows that the G_δ-

diagonal property is equivalent to a weak form of developability ([41], Theorem 2.2). A space X has a G_δ-diagonal if and only if there exists a sequence $(\mathcal{G}_n)$ of open covers of X such that for each $x, y \in X$ with $x \neq y$, there exists $n \in \omega$ with $y \notin st(x, \mathcal{G}_n)$ (equivalently, for each $x \in X, x = \bigcap_n st(x, \mathcal{G}_n)$).

A space X is said to have a regular G_δ-diagonal if the diagonal $\Delta = \{(x, x) : x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of Δ in $X \times X$. Every space with a regular G_δ-diagonal is Hausdorff. Indeed, according to Zenor ([110]), a space X has a regular G_δ-diagonal if and only if there exists a sequence $\{\mathcal{V}_n : n \in \omega\}$ of open covers of X with the following property:

For any two distinct points x and y in X, there are open neighborhoods O_x and O_y of x and y, respectively, and $k \in \omega$ such that no element of $\mathcal{V}$ intersects both O_x and O_y.

A space X has a G^*_δ-diagonal if there exists a sequence $(\mathcal{G}_n)$ of open covers such that for each $x \in X, \{x\} = \bigcap_n \overline{st(x, \mathcal{G}_n)}$.

Clearly a space has a regular G_δ-diagonal implies that it has a G^*_δ-diagonal and a space has a G^*_δ-diagonal implies that it has a G_δ-diagonal.

Definition 1.1.8. ([3]) A Tychonoff space X is a *p-space* if there exists a sequence $\{\mathscr{U}_n : n \in \omega\}$ of families of open subsets of the Stone-Čech compactification βX such that

a) each $\mathscr{U}_n$ covers X for each $n \in \omega$;

b) $\bigcap_{n\in\omega} \mathrm{st}(x, \mathscr{U}_n) \subset X$ for each $x \in X$, where $\mathrm{st}(x, \mathscr{U}_n) = \bigcup\{U \in \mathscr{U}_n : x \in U\}$.

The following is an internal characterization of p-spaces.

Theorem 1.1.9. ([27]) A space X is a p-space if and only if there exists a sequence $(\mathcal{G}_n)$ of open covers of X satisfying the following condition: If for each n, $x \in G_n \in \mathcal{G}_n$ then

(i) $\bigcap_n \overline{G_n}$ is compact;

(ii) $\{\bigcap_{i\leqslant n} \overline{G_i} : n \in \omega\}$ is an outer network for the set $\bigcap_n \overline{G_n}$ i.e., every open set containing $\bigcap_n \overline{G_n}$ contains some $\bigcap_{i\leqslant n} \overline{G_i}$.

Proof. Necessity. Let X be a p-space and $\mathcal{U}_n$ a sequence of open covers of βX satisfying a) and b) in Definition 1. Then there exists a sequence $\{\mathcal{G}_n : n \in \omega\}$ of open covers of X satisfying: For each $n \in \omega$ and $G \in \mathcal{G}_n$ there exists $U \in \mathcal{U}_n$ such that $cl_{\beta X}(G) \subseteq U$. We show that $\{\mathcal{G}_n : n \in \omega\}$ is a sequence of open covers of X satisfying the conditions (i) and (ii). For $x \in X$ and the sequence $\{G_n : n \in \omega\}$ of open subsets in X with $x \in G_n \in \mathcal{G}_n$ put $C_x = \bigcap_n \overline{G}_n$. Then

(i) C_x is compact, since $\bigcap_{n\in\omega}(cl_{\beta X} G_n) \subseteq \bigcap_{n\in\omega} st(x, \mathcal{U}_n) \subseteq X$ and $C_x = \bigcap_{n\in\omega}(X \cap cl_{\beta X} G_n) = \bigcap_{n\in\omega}(cl_{\beta X} G_n)$.

(ii) Let $C_x \subseteq G \in \tau(X)$. Take $U \in \tau(\beta X)$ such that $G = U \cap X$. Then $\{U\} \cap \{\beta X \setminus cl_{\beta X} G_n\}$ is an open cover of βX. Then there exists $k \in \omega$ such that $\bigcap_{n\leqslant k} cl_{\beta X} G_n \subseteq U$. It implies that $\bigcap_{n\leqslant k} \overline{G}_n \subseteq G$ which shows that $\{\bigcap_{i\leqslant n} \overline{G}_i : n \in \omega\}$ is an outer network for the set C_x.

Sufficiency. Let $\{\mathcal{G}_n : n \in \omega\}$ be a sequence of open covers of X satisfying the conditions (i) and (ii). Expand $\{\mathcal{G}_n : n \in \omega\}$ to $\mathcal{U}_n = \{U_G : G \in \mathcal{G}_n\}$, where U_G is open in βX and $U_G \cap X = G$. Thus $\{\mathcal{G}_n : n \in \omega\}$ covers X. Let $x \in X$. Assume that there exist $y \in \bigcap_{n\in\omega} st(x, \mathcal{U}_n) \setminus X$. Then there exists a sequence $\{U_n : n \in \omega\}$ of open subset in βX such that $\{x, y\} \subseteq U_n \in \mathcal{U}_n$. Since $\bigcap_{n\in\omega} \overline{U_n \cap X}$ is compact in X, there exists $U \in \tau(\beta X)$ and $k \in \omega$ such that

$$\bigcap_{n\leqslant k} \overline{U_n \cap X} \subseteq U \subseteq cl_{\beta X} \subseteq \beta X \setminus \{y\}.$$

Put $W = (\bigcap_{n\leqslant k} U_n) \cap (\beta X \setminus cl_{\beta X} U)$. Clearly, $W \cap X = \varnothing$. But $y \in W$ and X is dense in βX, a contradiction. □

A completely regular space X is *Čech-complete* if X is a G_δ-subset in some compactification of X.

Chapter 2

Generalized metrizable properties and cardinal invariants in paratopological and semitopological groups

In this chapter, we consider the generalized metrizable properties on paratopological groups and semitopological groups. We introduce the method of "g-function"in paratopological groups and quasitopological groups and the submetrizability of paratopological and semitopological groups is considered. Moreover, we give some examples to show that some theorems on the mappings between topological groups can not be extended to paratopological groups.

2.1 First-countable paratopological groups and quasitopological groups

In this section, we characterize first-countable paratopological and quasitopological groups and consider the G_δ-diagonal of them. Moreover, we consider developable and quasi-developable paratopological (quasitopological) groups.

2.1.1 First-countable paratopological groups

It is well known that a topological group is metrizable if and only if it is first-countable ([23] [45]). Recall that a function $d : X \times X \to \mathbb{R}^+$ is called a quasi-metric on the set X if for each $x, y, z \in X$ satisfying that (i) $d(x, y) = 0$ if and only if $x = y$; (ii) $d(x, z) \leqslant d(x, y) + d(y, z)$. A topological space X is said to be quasi-metrizable if there is a quasi-metric on X such that $\{B(x, \varepsilon) : \varepsilon > 0\}$ forms a base at each $x \in X$. In 2001, Ravsky ([77], Proposition 3.1) proved that a paratopological group is quasi-metrizable if and only if it is first-countable. Recently, Liu C and Lin S ([62], Propositon 2.1) proved the same result with the method of "g-function". A g-function on a topological space (X, τ) is a mapping $g : \omega \times X \to \tau$ such that $x \in g(n, x)$ for each $n \in \omega$. A number of generalized metric spaces have been defined or characterized with g-function, which generalized ball neighborhoods in metric spaces. A Hausdorff space (X, τ) is quasi-metrizable if and only if there is a function $g : \omega \times X \to \tau$ such that (i) $g(n, x) : n \in \omega$ is a local base at x; (ii) $y \in g(n+1, x) \Rightarrow g(n+1, y) \subseteq g(n, x)$ ([41], Theorem 10.2). In the following, we gave the proof of Liu C and Lin S.

Proposition 2.1.1. ([62], Proposition 2.1) *Every Hausdorff first-countable paratopological group is quasi-metrizable.*

Proof. Suppose (X, τ) is a first-countable paratopological group. Let $\{V_n : n \in \omega\}$ be a countable local base at the neutral element e such that $V_{n+1}^2 \subsetneq V_n$. Define $g : \omega \times X \to \tau$ as follows: $g(n, x) = xV_n$ for each $n \in \omega$ and $x \in X$. It is obvious that $\{g(n, x) : n \in \omega\}$ is a local base at x. Suppose $y \in g(n+1, x)$, then $y \in xV_{n+1}, y = xv_1$ for some $v_1 \in V_{n+1}$. Take $z \in g(n+1, y)$, then $z = yv_2$ for some $v_2 \in V_{n+1}$. Thus $z = yv_2 = xv_1v_2 \in xV_{n+1}V_{n+1} \subseteq xVn = g(n, x)$. By ([41], Theorem 10.2), X is quasi-metrizable. □

In the following, we consider the G_δ-diagonal property in paratopological and semitopological groups. Unlike topological groups, a first-countable paratopological group might not be metrizable. But they also have some important generalized metrizable properties. In 1999, it was proved by Chen Y Q ([32]) that every Hausdorff first-countable paratopological group has a G_δ-diagonal. This was then extended to semitopological groups by Arhangel'skiĭ A V and

Reznichenko E A([16], Corollary 2.5). They showed that a Hausdorff semitopological group of countable π-character has a G_δ-diagonal. The following result was established by Arhangel'skiĭ A V and Burk D K in ([12]) for Abelian paratopological groups and by Liu C in the general case.

Proposition 2.1.2. ([60], Theorem 2.1) *Every first-countable Hausdorff paratopological group has a regular G_δ-diagonal.*

Proof. Fix a countable base $\{V_n : n \in \mathbb{N}\}$ at the neutral element e in G with $V_{n+1}^2 \subseteq V_n$. Let $x \in G$, then xV_n, V_nx are open for $n \in \mathbb{N}$ since G is a paratopological group. For $x \in G, n \in \mathbb{N}$, let $W_n(x) = xV_n \cap V_nx$. Then $W_n(x)$ is an open neighborhood of x. Let $\mathcal{G}_n = \{W_n(x) : x \in G\}$ for $n \in \mathbb{N}$. Then $\{\mathcal{G}_n : n \in \mathbb{N}\}$ is a sequence of open coverings of G.

By Zenor's characterization of regular G_δ-diagonal, we only prove the following claim.

Claim. For $y, z \in G, y \neq z$, there is $k \in \mathbb{N}$ such that no element of $\mathcal{G}_k$ intersects both yV_k and zV_k.

Suppose not; for any $n \in \mathbb{N}$, there is an element $W_n(x_n) \in \mathcal{G}_n$ such that $yV_n \cap W_n(x_n) \neq \varnothing$ and $W_n(x_n) \cap zV_n \neq \varnothing$. Then there are a_n, b_n, c_n, d_n and f_n in V_n such that $ya_n = x_nb_n, x_nc_n = d_nx_n = zf_n, ya_n = d_n^{-1}d_nx_nb_n = d_n^{-1}zf_nb_n$. Since $a_n \to e$, we have $ya_n \to y$, hence $d_n^{-1}zf_nb_n \to y$. It is clear that $d_n \to e$ since $d_n \in V_n$. Since G is a paratopological group, then $d_nd_n^{-1}zf_nb_n \to ey = y$, hence $zf_nb_n \to y$. Notice that $f_n, b_n \in V_n$, thus $f_nb_n \to e$, hence $zf_nb_n \to z$. G is Hausdorff, then $y = z$, this is a contradiction.

Therefore, G has a regular G_δ-diagonal. □

The property of having a regular G_δ-diagonal is strictly stronger than that of having a G_δ-diagonal. This is witnessed, for example, by the fact that every space of countable cellularity and with a regular G_δ-diagonal has cardinality at most 2^ω ([30], Theorem 2.2). It is also shown in ([11], Corollary 6) that every first-countable Hausdorff paratopological group with cellularity $\leqslant 2^\omega$ has cardinality less than or equal to 2^ω.

Let $\mathcal{U}$ be a collection of subsets of a space X. The star of $\mathcal{U}$ with respect to $A \subseteq X$, denoted by $st(A,\mathcal{U})$, is the set $\bigcup\{U \in \mathcal{U} : U \cap A \neq \varnothing\}$. When $A = \{x\}$, we simply write $st(x,\mathcal{U})$. We put $st^1(x,\mathcal{U}) = st(x,\mathcal{U})$ and recursively define

$st^{n+1}(x,\mathcal{U}) = st(st^n(x,\mathcal{U}),\mathcal{U})$. Let n be a positive integer. We say that a space X has a G_δ-diagonal of rank n if there exists a countable collection $\{\mathcal{U}_k : k \in \mathbb{N}\}$ of open covers of X such that $\bigcap\{st^n(x,\mathcal{U}_k) : k \in \mathbb{N}\} = \{x\}$ for each $x \in X$. If a space has a G_δ-diagonal of any possible rank, then we say that it has a G_δ-diagonal of infinite rank. Zenor has pointed out that in [110] that a diagonal of rank 3 is always a regular G_δ-diagonal.

The result established in Proposition 2.1.2 was extended to countable π-character by Sánchez I ([85], Theorem 2.25) who showed that every Hausdorff paratopological group G with countable π-character has a regular G_δ-diagonal. Moreover, it is showed by Arhangel'skiĭ A V and Bella A ([11]) that very Hausdorff first-countable paratopological group has a G_δ-diagonal of infinite rank. More recently, Sánchez I ([86]) showed if a dense subgroup H of a Hausdorff paratopological group G such that H has countable π-character, then G has a G_δ-diagonal of infinite rank.

A semitopological group G is called *ω-narrow* if, for every open neighborhood V of the neutral element e of G, there exists a countable subset A of G such that $AV = G = VA$. In the following, we discuss ω-narrow paratopological groups.

Theorem 2.1.3. ([51]) *Every first-countable ω-narrow semitopological group is separable.*

Proof. Let G be a first-countable ω-narrow semitopological group and $\{U_n : n \in \mathbb{N}\}$ a local base at the neutral element $e \in G$.

First we construct a countable subgroup H of G such that for each n, $HU_n = G$. For each $n \in \mathbb{N}$, take a countable subset $A_n \subseteq G$ such that $A_nU_n = G$. Put $C_n = A_n \cup A_n^{-1} \cup \{e\}$ and $C = \bigcup_{n=1}^{\infty} C_n$. Then C_n and C are countable symmetric subsets of G. Put $H = \bigcup_{k=1}^{\infty} C^k$. Then H is a countable subgroup of G satisfying $HU_n = G$ for each $n \in \mathbb{N}$.

We claim that H is dense in G. To see this, let U be an arbitrary non-empty open subset of G. Fix a point $x \in U$. Then there exists an open neighborhood U_n of e such that $U_nx \subseteq U$. Since $HU_n = G$, we have $HU_nx = Gx = G$. Hence, $(HU_nx) \cap H \neq \varnothing$. Take h_1, $h_2 \in H$ and $y \in U_n$ such that $h_1yx = h_2$. Then $yx = h_1^{-1}h_2 \in H$, since H is a subgroup of G. Hence, $yx \in (U_nx) \cap H \subseteq U \cap H$. The proof is complete. □