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edited by

Pan-Tai Liu and Emilio Roxin

DIFFERENTIAL GAMES AND CONTROL THEORY III

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Edited by

PAN-TAI LIU and EMILIO ROXIN

Department of Mathematics
University of Rhode Island
Kingston, Rhode Island

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PREFACE

The papers appearing in this volume were presented at the Third Kingston Conference on Differential Games and Control Theory held at the University of Rhode Island June 5-8, 1978.

About half of these papers deal with deterministic systems and the other half with stochastic systems. The contents of these papers range from theoretical analysis to practical techniques in the control of dynamic systems. It is hoped that this collection of papers on a diversity of topics will constitute a good reference book for researchers in this branch of applied mathematics which has been and continues to be an area of active research.

Another volume containing papers on applications of control theory to economics and management science that were presented at the Conference will be published in the near future.

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We would like to thank, in particular, Dr. Neal Glassman and Dr. Thomas Varley of the Office of Naval Research and Dr. William Ferrante, Vice President of the University of Rhode Island, for their support. Finally, we would like to express our deep appreciation to Mrs. Marguerite Ellis for her excellent typing of the manuscript.

Pan-Tai Lui
Emilio Roxin

LIST OF CONTRIBUTORS

AUSTIN BLAQUIÈRE, Laboratoire d'Automatique Théorique, Université de Paris, Paris, France

ROBERT J. ELLIOTT, Department of Mathematics, University of Kentucky, Lexington, Kentucky, and Department of Pure Mathematics, University of Hull, England

WENDELL H. FLEMING, Department of Mathematics and Lefschetz Center for Dynamical Systems, Division of Applied Mathematics, Brown University, Providence, Rhode Island

BERNHARD GOLLAN, Mathematisches Institüt der Universität Würzburg, Am Hubland, Würzburg, West Germany

W. A. GRUVER, Graduate Program in Operations Research, North Carolina State University, Raleigh, North Carolina

C. F. KLEIN,* Graduate Program in Operations Research, North Carolina State University, Raleigh, North Carolina

G. LEITMANN, University of California at Berkeley, Berkeley, California

J. LEVINE, Centre d'Automatique and Informatique de l'ENSMP, Fontainebleau, France

J. A. MATHEWS, Department of Computing and Control, Imperial College of Science and Technology, London, England

PAWEL J. SZABLOWSKI, Institute of Mathematics, Technical University of Warsaw, Warsaw, Poland

CHUN-PING TSAI, Department of Mathematics, University of Kansas, Lawrence, Kansas

J. WARGA, Department of Mathematics, Northeastern University, Boston, Massachusetts

*Currently with Bell Telephone Laboratories, Holmdel, New Jersey

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NECESSARY AND SUFFICIENCY CONDITIONS FOR OPTIMAL STRATEGIES IN IMPULSIVE CONTROL

Austin Blaquière

Laboratoire d'Automatique Théorique
Université de Paris 7
2 Place Jussieu
75005 Paris, France

§1. INTRODUCTION

To motivate the mathematical formulation of the problem we shall first present two examples.

§1.1 Optimal Maintenance and the Life of Machines [1]

Let us characterize the quality of service of a machine by a number x between zero and 1. The index x is a function of time t since the machine does get old and needs to be repaired. We shall suppose that during the life of a machine the evolution of x is defined by the equation

$$\dot{x} = -kx + mu$$

where $u = u(t)$ is the rate of maintenance expenses, measured in monetary units per unit of time, and k and m are positive constants.

Also we shall suppose that the total profit in a planning period, say $0 \leq t \leq T$, is

$$W(T) = \int_0^T (Ax(t) - u(t))dt + \sum_{i=1}^{N(T)} (Kx(t_i) - C)_i$$

N(T) being the number of times (and t_i, i = 1, ..., N(T), the times at which) we decide to stop the use of the machine, to sell it, and to purchase a new one. A, K, and C are positive constants.

Here we have supposed that, for a quality x,

1. the net operating receipt per unit of time is Ax;
2. every time one decides to resell a machine, the sales value is Kx;
3. the investment cost of a new machine is C.

Maintenance expenditures are assumed to be subject to the constraint

$$0 \le u(t) \le \bar{u}$$

where $\bar{u}$ is a constant upper limit on maintenance expenditures.

We want to determine the times at which a machine must be replaced by a new one, and the rate of maintenance expenses, so as to maximize the total profit in the planning period.

§1.2 Optimal Repainting Plan for a Roadside Inn [5]

The profit of the owner of a roadside inn, on some prescribed interval of time $0 \le t \le T$, is a function of the number of strangers who pass by on the road each day, and on the number of times the inn is repainted during that period. The ability to attract new customers into the inn depends on its appearance, which is supposed to be indexed by a number x.

During time intervals between paint jobs, x decays according to the law

$$\dot{x} = -kx \qquad k = \text{constant} > 0$$

The total profit in the planning period $0 \leq t \leq T$ is supposed to be

$$W(T) = A \int_0^T x(t)dt - \sum_{i=1}^{N(T)} C_i$$

where $N(T)$ is the number of times the inn is repainted, C_i, $i = 1, \ldots, N(T)$, the cost of each paint job, and A a strictly positive constant. The owner of the inn wishes to maximize his total profit.

Now let us come to the mathematical formulation of the problem.

§2. PROBLEM STATEMENT

§2.1 Strategies and Paths

We shall be concerned with a *dynamical system* defined by its *state*, a set of n real numbers, $x = (x_1, \ldots, x_n) \in R^n$, which change in a prescribed manner with the passing of time t. We shall suppose that the state lies in some domain X of R^n, and that one of its components, say x_n, is time t. The evolution of the state is influenced, or as we say controlled, by an agent J_0 through his choice of a *strategy* s in a prescribed *strategy set* S_0. First, we shall define this strategy set and the rule that governs the motion of the state. Let U, M be prescribed nonempty open subsets of R^{d_1} and R^{d_2}, respectively. Let K_u and K_μ be prescribed nonempty subsets of U and M, respectively. Let P and Π be prescribed sets of functions defined on X with range in K_u and K_μ, respectively.

Let Δ be the collection of all closed subsets of X in the topology induced by R^n on X.

<u>Definition 1</u> The strategy set of J_0 is $S_0 = \Delta \times P \times \Pi$.

Now let f(•) and g(•) be prescribed functions of class C^1 on $X \times U$ and $X \times M$, respectively; namely

$$f(\cdot): \quad X \times U \to R^n \qquad (x,u) \to f(x,u)$$
$$g(\cdot): \quad X \times M \to R^n \qquad (x,\mu) \to g(x,\mu)$$

with $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$, $f_n(x,u) \equiv 1$, and $g_n(x,\mu) \equiv 0$.

<u>Definition 2</u> A strategy s, $s = (Y, p(\cdot), \pi(\cdot)) \in S_0$ is *admissible* if and only if

$$x \in Y \Rightarrow x + g(x,\pi(x)) \in X - Y$$

Let S be the set of all admissible strategies.

<u>Definition 3</u> A function $x(\cdot)$: $[t_0,t_1] \to \bar{X}$†, defined on some interval of time $[t_0,t_1]$ is a *path in* R^n *generated by* s, $s = (Y, p(\cdot), \pi(\cdot)) \in S$, *from initial state* $x^0 \in X$ if and only if

1. $x(t_0) = x^0$
2. $x(\cdot)$ is piecewise continuous on $[t_0,t_1]$--let $T[t_0,t_1]$ denote the set of its discontinuity points
3. $x(t) = x(t - 0)$ for $t \neq t_0$
4. $t \in T[t_0,t_1] \Rightarrow x(t) \in Y$ and $x(t + 0) = x(t) + g(x(t), \pi(x(t)))$

---

†$\bar{X}$ denotes the closure of X in the topology of R^n.

5. for all t which are not discontinuity points, except possibly at $t = t_1$, $x(t) \in X - Y$
6. $x(\cdot)$ is differentiable, and $dx(t)/dt = f(x(t), p(x(t)))$ for all $t \in [t_0, t_1]$, except on an at most denumerable subset of $[t_0, t_1]$.

It follows from 3 that $t_1 \notin T[t_0, t_1]$.

§2.2 Playable Strategies and Terminating Paths

We shall suppose that J_0 desires to steer the state from a given initial state x^0 to a state belonging to a prescribed *target* set, $\theta \subset \partial X$.

Definition 4 A path $x(\cdot) : [t_0, t_1] \to \bar{X}$ generated by $s \in S$ from initial state $x^0 \in X$ is a *terminating path* if and only if $x(t_1) \in \theta$.

Definition 5 A strategy $s \in S_0$ is *playable at* x^0, $x^0 \in X$, if and only if it is admissible and it generates a terminating path from x^0.

Let $J(x^0)$ denote the set of all strategies playable at x^0 and $I(x^0, s)$ the set of all terminating paths generated from x^0 by playable strategy $s \in J(x^0)$.

§2.3 Cost of a Path

Let $f_0(\cdot)$ and $g_0(\cdot)$ be prescribed functions of class C^1 on $X \times U$ and $X \times M$, respectively; namely

$$f_0(\cdot): \quad X \times U \to R \qquad (x,u) \to f_0(x,u)$$

$$g_0(\cdot): \quad X \times M \to R \qquad (x,\mu) \to g_0(x,\mu)$$

Let $\tilde{\theta}_0(\cdot)$ be a prescribed function of class C^1 on some domain $\mathcal{D} \subset R^n$ containing θ:

$$\tilde{\theta}_0(\cdot): \mathcal{D} \to R \qquad x \to \tilde{\theta}_0(x)$$

and let $\theta_0(\cdot) : X \cup \theta \to R$ coincide with $\tilde{\theta}_0(\cdot)$ on θ, and $\theta_0(x) = 0$ for all $x \in X$.

From now on we shall suppose that the set S of admissible strategies is such that, for all $x^0 \in X$, all $s \in S$, and for all paths $x(\cdot)$ emanating from x^0 generated by s, the following integral is defined:

$$\int_{t_0}^{t_1} f_0(x(t),p(x(t)))dt$$

where $[t_0,t_1]$ is the interval on which $x(\cdot)$ is defined.

The cost of a path $x(\cdot) : [t_0,t_1] \to \bar{X}$ generated by $s = (Y,p(\cdot),\pi(\cdot)) \in S$ from initial state $x^0 \in X$ is

$$V(x^0,s,x(\cdot)) = \theta_0(x(t_1)) + \int_{t_0}^{t_1} f_0(x(t),p(x(t)))dt + \sum_{t \in T[t_0,t_1]} g_0(x(t),\pi(x(t)))$$

§2.4 Optimality of a Strategy

Let us suppose that J_0 desires to minimize the cost of transfer from an initial state to the target.

Definition 6 A strategy s^* is *optimal* at $x^0 \in X$ if and only if

1. s^* is playable at x^0
2. $V(x^0,s^*,x^*(\cdot)) \le V(x^0,s,x(\cdot))$ for all $s \in J(x^0)$, all $x(\cdot) \in I(x^0,s)$, and all $x^*(\cdot) \in I(x^0,s^*)$