North-Holland

Computational and Combinatorial Methods in Systems Theory

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COMPUTATIONAL AND COMBINATORIAL METHODS IN SYSTEMS THEORY

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1986

NORTH-HOLLAND
AMSTERDAM · NEW YORK · OXFORD · TOKYO

ISBN: 0 444 70031 5

Publishers:
ELSEVIER SCIENCE PUBLISHERS B.V.
P.O. Box 1991
1000 BZ Amsterdam
The Netherlands

Sole distributors for the U.S.A. and Canada:
ELSEVIER SCIENCE PUBLISHING COMPANY, INC.
52 Vanderbilt Avenue
New York, N.Y. 10017
U.S.A.

Library of Congress Cataloging-in-Publication Data

Computaticnal and combinatorial methods in systems theory.

Selected papers originally presented at the 7th International Symposium on the Mathematical Theory of Networks and Systems, held at the Royal Institute of Technology in Stockholm, June 10-14, 1985.
Includes index.
1. Control theory--Congresses. 2. Numerical analysis --Congresses. 3. Combinatorial analysis--Congresses. I. Byrnes, Christopher I., 1949- . II. Lindquist, Anders. III. MTNS International Symposium (7th : 1985 : Kungl. Tekniska högskolan)
QA402.3.C575 1986 629.8'312 86-9041
ISBN 0-444-70031-5 (U.S.)

PRINTED IN THE NETHERLANDS

COMPUTATIONAL AND COMBINATORIAL METHODS
IN SYSTEMS THEORY

PREFACE

In classical mathematics the problems of computation and approximation occupied a central position in the application and the development of powerful new theories and there can be no doubt that the importance of these twin themes will endure for the foreseeable future. Systems theory, depending as it does on a variety of mathematical disciplines for its basic tools, provides a natural source of interesting problems for those classical and modern applications of mathematics.

In this volume, we present a collection of papers on both the discrete and the continuous aspects of these themes, ranging from fundamental systems theoretic applications of interpolation theory, numerical linear algebra and computational complex analysis to the development of programming languages and strategies for flexible manufacturing systems and VLSI design. The papers of this volume, which attest to the vitality and importance of this cross-fertilization between pure and applied, were selected from the invited and contributed papers presented at the 7th International Symposium on the Mathematical Theory of Networks and Systems held at the Royal Institute of Technology in Stockholm on June 10-14, 1985.

We would like to take this opportunity to thank the following research agencies for their generous support of MTNS-85: the National Swedish Board for Technical Development (STU), the Office of Naval Research Branch Office, London (ONRL), the Research Institute of National Defence (FOA), the Swedish Institute for Applied Mathematics (ITM), the Swedish Natural Science Research Council (NFR) and the USAF European Office of Aerospace Research and Development (EOARD).

Christopher I. Byrnes and Anders Lindquist
Stockholm, January 29, 1986.

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Computational and Combinatorial Methods in Systems Theory
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ALGEBRAIC PROBLEMS ARISING IN ROBUST STABILIZATION AND COMPENSATION*

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The problem of designing a low order robust stabilizing compensator for a linear time invariant system is described and formulated in algebraic terms as a static output feedback stabilization problem. A numerical approach to this problem based on a suitably formulated optimization problem is presented. This is an extension of the recently introduced algorithm for pole assignment via Sylvester's equation. Some examples are given.

INTRODUCTION

The problem of feedback stabilization of a given system by a compensator is the central problem of control theory and design and almost every major theoretical tool developed in the control literature is directed towards this problem. Among the main distinct approaches to this problem are the LQG theory, the state feedback observer approach, and the Brasch-Pearson pole placement approach which are now standard and may be found in textbooks such as [1].

From a practical point of view none of the above approaches are adequate or even reasonably satisfactory. The reason for this is twofold:

1) the dynamic order of the controllers that result in any of the above methods is very high, and
2) the controllers are not robust, i.e., do not provide protection against plant parameter perturbations.

The problem of high order is serious because typically one is interested in low order controllers that are capable of controlling high order plants rather than vice versa. The control system designer who is handed a high order controller as the solution to the stabilization problem faces a high dimensional parameter space in which he has to carry out adjustments to meet various conflicting design specifications. This poses formidable conceptual and computational problems which are best avoided if at all possible.

The problem of robustness is well known in the literature: the controller must preserve stability of the closed loop system when the plant undergoes perturbations from a specified class. However, despite the extensive literature on the subject, the only result that gives a constructive synthesis procedure for generating a robust controller is, to the best of our knowledge, the recent result of Kimura [2]. The result of [2] is restricted to single input single

*This research was supported by the National Science Foundation under Grant No. ECS-8309792

output plants, considers additive perturbations of the transfer function, and in general will provide high order solutions. With regard to the LQG approach it is now well known that the good stability margins provided by the LQ solutions disappear when implemented via output feedback.

In the present paper we attempt to provide a reasonably realistic formulation of the problem of designing a low order robust stabilizing compensator. This algebraic formulation is motivated by the observation that every compensator corresponds to a static output feedback controller for an augmented system derived from the plant. Robust compensator design then corresponds to the design of a robust static output feedback control law. It is proposed that analytical approaches be developed by researchers for the resolution of this unsolved problem. A numerical approach for solution of this problem will be described along with examples.

PROBLEM FORMULATION

Let

$$\mathcal{S}_p : \begin{aligned} \dot{x} &= Ax + Bu, \; x \in \mathbb{R}^n \\ y &= Cx \end{aligned}$$

denote the plant and

$$\mathcal{S}_p : \begin{aligned} \dot{z} &= A_c z + B_c u_c, \; z \in \mathbb{R}^q \\ y_c &= C_c z + D_c u_c \end{aligned}$$

the controller with the feedback connection

$$u_c = y$$

$$y_c = u$$

so that the closed loop system equations are:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \underbrace{\begin{pmatrix} A+BD_cC & BC_c \\ B_cC & A_c \end{pmatrix}}_{A_{c\ell}} \begin{pmatrix} x \\ z \end{pmatrix}$$

The problem of feedback stabilization is:

Given the plant (A, B, C) find a controller (A_c, B_c, C_c, D_c) so that the eigenvalues of $A_{c\ell}$ lie in the left half of the complex plane, i.e., $\sigma(A_{c\ell}) \subset \mathbb{C}^-$.

The necessary and sufficient conditions for the existence of some controller is given by stabilizability and detectability of (A,B,C) and under this assumption the problem can be reformulated by considering (A,B,C) to be controllable and observable, without loss of generality [1]. When the order q of the controller is fixed, however, no necessary and sufficient conditions for the existence of (A_c, B_c, C_c, D_c) are known.

Now since

$$A_{cl} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}}_{A_q} + \underbrace{\begin{bmatrix} B & 0 \\ 0 & I_q \end{bmatrix}}_{B_q} \underbrace{\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}}_{K_q} \underbrace{\begin{bmatrix} C & 0 \\ 0 & I_q \end{bmatrix}}_{C_q}$$

$$= A_q + B_q K_q C_q$$

it is clear that the problem of feedback stabilization by a fixed order controller is equivalent to stabilizing $A_q + B_q K_q C_q$ by choice of K_q. The minimal order of a stabilizing controller is the smallest integer q^* such that there exists K_{q^*} for which $A_{q^*} + B_{q^*} K_{q^*} C_{q^*}$ is stable.

The problem of finding q^* is one of the outstanding unsolved problems of control theory. The available bounds on q^* are

1) the pole-placement result of Brasch-Pearson [3] which states that

$$q^* \leq \text{Min} \{P_c, P_o\}$$

where P_c is the controllability index of (A,B) and P_o is the observability index of (C,A), and

2) the stabilization result of Kimura [4] according to which

$$q^* \leq n + 1 - r - m$$

where r and m are the number of inputs and outputs respectively.

In the following section we present a numerical approach to the output feedback stabilization problem. This algorithm can be applied to the problem of finding a low order stabilizing compensator by sequentially applying it to (A_i, B_i, C_i) for i = 0,1,2 ... until a q is found such that $A_q + B_q K_q C_q$ is stable for some K_q. The algorithm also provides for some robustness in the sense that a K is found that attempts to orthogonalize the eigenvector set of A + BKC. It is well known [5] and [6] that such a choice of K causes the eigenvalues of A + BKC to be relatively insensitive to first order perturbations in the entries of the matrix.

NUMERICAL ALGORITHM

In this section we display an algorithm for finding K, if it exists, so that A + BKC is stable. We shall extend the state feedback pole assignment algorithm via Sylvester's equation, developed in [7], to handle this problem. The advantage of this algorithm is that it parametrizes the entire family of solutions to the state feedback problem so that an efficient approximation to the output feedback problem may be obtained.

The state feedback pole assignment algorithm of [7] is:

1) Solve for X in

$$AX - X\tilde{A} = -BG$$

for some $\tilde{A}$ with $\sigma(\tilde{A}) = \Lambda$ and for any G so that $(G,\tilde{A})$ is observable. Then by the result of [8] X will almost always be nonsingular.

2) Solve for F in

$$FX = G$$

Then $\sigma(A + BF) = \Lambda$.

To adapt this to the output feedback case we also want that F be approximated by KC, i.e., $\| F-KC \|$ be small. Moreover since the closed loop eigenvalues are those of A + BKC it is important to ensure that $\sigma(A + BKC)$ and $\sigma(A + BF)$ be "close" if F and KC are "close." This can be taken into consideration by attempting to orthonormalize the eigenvector set of A + BF. Finally the assigned eigenvalues are required to lie in a region $\Omega \subset C^-$ which is appropriately chosen. Based on these consideration we formulate the optimization problem:

Find F and K so that

$$J = \alpha_1 \text{trace}\, [I-XX^T]^2 + \alpha_2 \text{trace}\, (F-KC)^T (F-KC)$$ is minimized where $\sigma(A+BF) \subset \Omega$ and X is the eigenvector matrix of A + BF.

As mentioned before we may take $\sigma(\tilde{A}) \subset \Omega$ with $\tilde{A}$ diagonal so that in

$$AX - X\tilde{A} = -BG$$
$$FX = G$$

X becomes the eigenvector matrix of A + BF. The gradient of J with respect to G may be calculated. The details of this calculation are omitted and may be found in [9]. The result derived in [9] is that

$$\frac{\Delta J}{\Delta G} = 2 \quad \{\alpha_1 F \,[I - 2RC + (RC)^T RC](X^T)^{-1} - B^T U^T\}$$

where U satisfies

$$\tilde{A}U - UA = \alpha_1 X^{-1}[I + 2\,(RC)^T + (RC)(RC)^T]F^T F + 2\,\alpha_2\,[I-X^T X]X^T$$

and $R = C^T (CC^T)^{-1}$.

Based on this we may formulate the gradient based algorithm:

Step 0: Pick $\sigma(\tilde{A}_o) \subset \Omega$ and G_o

Step 1: Find F and K that minimizes

$$J = \alpha_1 \text{Trace}\{I - XX^T\}^2 + \alpha_2 \text{Trace}\{(F - KC)^T (F - KC)\}$$

with respect to G.

Step 2: IF $\sigma(A + BKC) \subset \Omega$
THEN stop (controller found)
ELSE continue to Step 3.

Step 3: Compute $\frac{\partial J}{\partial A}$ by numerical procedures

IF $\left\| \frac{\partial J}{\partial A} \right\| > \varepsilon$
THEN continue to Step 4.
ELSE stop

Step 4: Obtain a new $\tilde{A}_{i+1}$ using a gradient based method. Set i = i + 1 and continue to Step 1.

EXAMPLES

All examples shown here indicate that lower order output feedbacks were achieved and the region Ω was defined as the whole left half complex plane to achieve a stabilizing controller.

Example 1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$C = (1 \quad 0.5 \quad 1.5)$$

Initial guess of $G = \begin{bmatrix} 1 & 0.5 & 1.5 \\ -0.5 & -1 & 1 \end{bmatrix}$

Eigenvalues of A = (16.1168, -1.1168, 0).

Initial eigenvalues of $\tilde{A}$ = (-1,-2,-3).

Initial cost J_o = 659.42478159.

Final cost J_{opt} = 6.3778387856.

Obtained K = (-0.7538303194 -9.560368516)

Eigenvalues of A + BKC = $-2.402794 \pm j4.94279$, -1.19972433.

Example 2

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 1 & 0.5 & 0 \end{bmatrix}$$

Initial guess of $G = \begin{bmatrix} -5 & 2 & 2 & 1 \\ -1 & -1 & 2 & 1 \end{bmatrix}$

Eigenvalues of A = (1,0,0,0).

Initial eigenvalues of $\tilde{A}$ = $(-1 \pm j2, -2 \pm j1)$.

Initial cost of J_o = 159.758483.

Final cost J_{opt} = 3.492771.

$$\text{Obtained } K = \begin{bmatrix} -1.559706 & -1.225891 \\ 2.355928 & -10.53672 \end{bmatrix}$$

Eigenvalues of A + BKC = (−1.183542 ± j1.392888, −3.230272, −1.790421).

Example 3

$$A = \begin{bmatrix} 1 & 0.5 & 0.2 & 0 & 0 \\ 1 & -1 & 0 & -1.5 & 0 \\ 0 & 0 & -1.5 & 0.5 & 1 \\ 0 & 0 & 1.5 & 2 & 1.5 \\ 0 & 0 & 0 & -1.5 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0.5 & -1 & 0 & 1 \\ -1 & 0 & -0.5 & 1.5 & 0 \end{bmatrix}$$

$$\text{Initial guess of } G = \begin{bmatrix} 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Eigenvalues of A = (−1.22474, 1.22474, −1.8336, 1.66682 ± j1.52788).

Initial eigenvalues of $\tilde{A}$ = (−2.5 ± j2.5, −3 ± j3, −3.5).

Initial cost J_o = 440.983766.

Final cost J_{opt} = 6.650208.

$$\text{Obtained } K = \begin{bmatrix} 253.10249 & -50.30768 \\ 25.604106 & -16.62546 \end{bmatrix}$$

Eigenvalues of A + BKC = (−1.7441 ± j9.2792, −5.5774 ± j2.3377, −0.4822).

CONCLUDING REMARKS

The problem of finding a low order if not the smallest order controller that stabilizes a system has been shown to be equivalent to finding the smallest q such that $A_q + B_q K_q C_q$ is stable for some K_q. This is an unsolved problem and deserves to be solved in an efficient manner. The determination of K_q so that $A_q + B_q K_q C_q$ remains stable for a class of perturbations in (A,B,C) is the problem of robust stabilization and deserves the attention of researchers. This paper has indicated a numerical approach to address some of these problems but obviously the whole problem is open for solution.

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Acknowledgement: The authors are grateful to L.H. Keel for generating the numerical examples.