FORMULAS FOR NATURAL FREQUENCY AND MODE SHAPE

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PREFACE

The purpose of this book is to provide a summary of formulas and principles on the vibration of structural and fluid systems. It is intended to be a reference book for engineers, designers, and students who have had some introduction to the theory of vibrations. However, anyone with a grasp of basic physics and an electronic calculator should have little difficulty in applying the formulas presented here.

Vibrations of structures have been known since man first heard wind ruffle the leaves of trees. Quantitative knowledge of vibrations was mostly limited to empirical descriptions of pendulums and stringed instruments until the development of calculus by Sir Isaac Newton and Gottfried Wilhelm Leibnitz in the late 1600's. The first instance in which the normal modes of continuous systems were determined involved the modes of a hanging chain, which were described in terms of Bessel functions by Daniel Bernoulli in 1732. The beauty and intricacy of modal patterns were actually visualized in 1787 when Ernst Chladni developed the method of placing sand on a vibrating plate. Mathematical description of vibrating plates proved more elusive. In 1809 Napoleon Bonaparte presented the Paris Institute of Science with the sum of 3000 francs to be given as a prize for a satisfactory mathematical theory of the vibration of plates. This prize was finally awarded in 1816 to Mademoiselle Sophie Germain, who first derived the correct differential equation but obtained erroneous boundary conditions. The theory of plate vibration was completed in 1850 by Gustav Kirchhoff. During the 1850's, calculus was applied to the vibration analysis of a number of practically important structural systems. This led to Lord Rayleigh's publication of *The Theory of Sound* (1st ed., 1877), which remains in print today. Lord Rayleigh, born John William Strutt, independently developed his own laboratory and devoted himself to science. His fellow countrymen must have thought him a bit odd as he investigated the vibration modes of their church bells.

A. E. H. Love's *A Treatise on the Mathematical Theory of Elasticity* (4th ed., 1926) and Horace Lamb's *The Dynamical Theory of Sound* (2nd ed., 1925) together with Rayleigh's *The Theory of Sound* form the basis of modern vibration analysis. Solutions developed in each of these books are presented here. During the early and middle 1900's, the techniques presented in these books were applied to increasingly complex systems. Sophisticated approximate techniques such as those employed by Stephen P. Timoshenko in *Vibration Problems in Engineering* (1st ed., 1928) also appeared during that period.

The advent of reliable electronic computers in the 1950's and the widespread installation of these computers in the early 1960's led to two new parallel paths for analyzing complex systems. First, the computer made it possible to generate approximate semi-closed form solutions which rely on classical solution techniques but with numerical evaluation of certain terms which cannot be expressed in closed form. Second, the development of large digital computers has made it feasible to simulate systems directly using finite element models. Today it is possible to simulate virtually any well-defined linear system on a large general purpose digital computer and obtain its natural frequencies, mode shapes, and response without resort to a theoretical treatment. Of course, the result is purely numerical, and physical insight into the nature of the solution must still be obtained through classical reasoning.

A vibration analysis generally follows four steps. First, the structure or system of interest is identified, its boundary conditions are estimated, and its interfaces with other systems are plotted. Second, the natural frequencies and mode shapes of the structure are determined by analysis or direct experimental measurement. Third, the time dependent loads on the structure are estimated. Fourth, these loads are applied to an analytical model of the structure to determine its response. The crucial steps in the vibration analysis are the identification of the structure and the determination of its natural frequencies and mode shapes.

The aim of this book is to provide formulas for the natural frequencies and mode shapes of a wide range of structures in easily used form so that the analyst can rapidly obtain his result without either searching the literature or spending the hours ordinarily required to successfully complete a finite element numerical simulation.

This book was written by searching through the cornucopia of solutions available in the literature for those solutions whose practicality and generality make them useful tools for the engineer or designer. In order to yield a compact volume, only those solutions which could be adequately presented in a relatively small space have been included. Chapters 1 through 5 present definitions, symbols, instruction in units, basic principles, and geometric properties of rigid structures. Chapter 6 is devoted to systems with finite number of degrees of freedom: the spring-mass systems. Chapter 7 considers the dynamics of cable systems. Chapters 8 through 12 present results for beams, curved beams, membranes, plates, and shells. Some practical information on the stress analysis of these structures is also included. Chapters 13 and 14 are devoted to vibration in fluid systems and the effect of a surrounding fluid on the vibration of structural systems. Chapter 15 reviews the finite element computer codes presently available for general vibration analysis. Chapter 16 presents data on the properties of materials which are useful as inputs for vibration analysis.

The solutions presented in this book span the technical literature from the second edition of Lord Rayleigh's *The Theory of Sound*, published in 1894, to the journals of 1978. While many of the solutions presented here can be traced to the pre-1930 volumes of Horace Lamb, A. E. H. Love, and Lord Rayleigh, the majority of the results in this book were generated after 1960. Notable among the more recent results are the cable solutions of H. Max Irvine, the multispan beam solutions of Daniel J. Gorman, the plate solutions of Arthur W. Leissa, and the cylindrical shell solutions of C. B. Sharma and D. J. Johns. The formats used in this book were adapted from those developed by Raymond J. Roark, and Chapters 11 and 12 were born in the compilations of Arthur W. Leissa. Those familiar with the work of these two fine analysts will recognize its reflection in this book.

The following individuals reviewed various chapters in this book:

I. T. Almajan
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This book is dedicated to the individuals who developed the solutions presented here; it is far more their creation than mine.

R.D.B.
La Jolla, California

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DEFINITIONS

Added Mass—The mass of fluid entrained by a moving structure as it vibrates in a fluid. The added mass of many slender structures is comparable to the mass of fluid displaced by the structure. The natural frequency of vibrations in the presence of added fluid mass is lower than that which would be observed in a vacuum. See Chapter 14.

Beam—A structure whose cross-sectional properties and deflection vary along only a single axis. A slender beam is a beam whose characteristic cross-sectional dimensions are much less than the span of the beam and the distance between vibration nodes; therefore, the inertia associated with local rotation is overshadowed by the inertia developed in displacement and the deformation due to shearing of the cross section is overshadowed by bending deformations.

Boundary Condition—A constraint applied to a structure independent of time. Boundary conditions can be classified as either geometric or kinetic. Geometric boundary conditions arise from geometric constraints. For example, the displacement of a structure at a joint pinned to a rigid wall is zero. Kinetic boundary conditions arise from force or moments applied to a structure; for example, a pinned joint permits free rotation, so the kinetic boundary condition at a pinned joint is zero moment. (See Pinned Boundary, Clamped Boundary, Free Boundary, and Sliding Boundary.)

Bulk Modulus of Elasticity—The ratio of the tensile or compressive stress, equal in all directions (i.e., hydrostatic pressure), to the change it produces in volume. $B = E/[3(1 - 2\nu)]$ for an isotropic elastic material, that is, a material whose properties are the same in all directions. (Definitions of symbols are given in Chapter 2.)

Cable—A massive string. A uniform, massive one-dimensional structure which can bear only tensile loads parallel to its own axis. The bending rigidity of cables is zero. Cables, unlike chains, may stretch in response to tensile loads.

Cable Modulus—The rate of change in the longitudinal stress (axial force over cross-sectional area) in a cable for a small unit longitudinal strain. If the cable is a solid elastic rod, the cable modulus will be equal to the modulus of elasticity of the rod material. If the cable is woven from fibers, the cable modulus will be less than the modulus of elasticity of the component fibers. Typically, the cable modulus of woven steel cables is about 50% of the modulus of elasticity of the steel fibers.

Center of Gravity—The point on which a body can be balanced. The sum over a body of all elements of mass multiplied by the distance from any axis through the center of gravity is zero. The center of gravity is also called the center of mass. (See Chapter 5.)

Centroid—The geometric center of a plane area. The sum over a plane area of all elements of area multiplied by the distance from any axis through the centroid is zero. (See Chapter 5.)

Chain—A uniform, massive one-dimensional structure which can bear only tensile loads parallel to its own axis. The bending rigidity of chains is zero. Chains, unlike cables, do not stretch in response to tensile loads.

Clamped Boundary—A geometric boundary condition such that the structure can neither displace nor rotate along a given boundary.

Concentrated Mass (Point Mass)—A point in space with finite mass but zero moment of inertia for rotation about its center of mass.

Damping—The ability of a structure to absorb vibrational energy. Damping can be generated within the material of the structure (material damping), by the fluid surrounding the structure (fluid damping), or by the impact and scraping at joints (structural damping).

Deformation—The displacement of a structure from its equilibrium position.

Density—The mass per unit volume of a material.

Elastic—A term applied to a material if deformations of the material increase linearly with increasing load without regard to the sign or magnitude of the load. Many real materials of structural importance are elastic for loads below the onset of yielding.

Free Boundary—A boundary along which no restraints are applied to a structure. For example, the tip of a freely vibrating cantilever is a free boundary.

Isotropic—A term applied to a material whose properties are unchanged by rotation of the axis of measurement. Only two elastic constants, the modulus of elasticity (E) and Poisson's ratio (ν), are required to completely specify the elastic behavior of an isotropic material.

Linear—A term applied to a structure or material if all deformations increase in proportion to the load without regard to the sign, magnitude, distribution, or direction of the load. Many structures of practical importance are linear for loads below a maximum linear limit. Nonlinear behavior in a structure is ordinarily due to either a material nonlinearity such as yielding or a geometric nonlinearity such as buckling.

Membrane—A thin, massive, elastic uniform sheet which can support only tensile loads in its own plane. A membrane may be flat like a drum head or curved like a soap bubble. A one-dimensional membrane is a cable. A massless one-dimensional membrane is a string.

Mode Shape (Eigenvector)—A function defined over a structure which describes the relative displacement of any point on the structure as the structure vibrates in a single mode. A mode shape is associated with each natural frequency of a structure. If the deflection of a linear vibrating structure in some direction is denoted by Y(x, t), where x is a point on the structure and t is time, then if the structure vibrates only in the k mode, the deflection can be written as

$$Y(x, t) = \tilde{y}_k(x)\, y_k(t),$$

where $\tilde{y}_k(x)$ is the mode shape, which is a function only of space, and $y_k(t)$ is a function only of time. If the structure vibrates in a number of modes, the total displacement is the sum of the modal displacements:

$$Y(x, t) = \sum_{i=1}^{N} \tilde{y}_i(x)\, y_i(t).$$

Modulus of Elasticity (Young's Modulus)–The rate of change of normal stress for a unit normal strain in a given material. The modulus of elasticity has units of pressure. For most materials, within the limits of linear elasticity, the modulus of elasticity is independent of the sign of the applied stress. Some materials, such as wood, have a directional modulus of elasticity.

Moment of Inertia of a Body–The sum of the products obtained by multiplying each element of mass within a body by the square of its distance from a given axis. (See Product of Inertia of a Body and Chapter 5.)

Moment of Inertia of a Section–The sum of the products obtained by multiplying each element of area within a section by the square of its distance from a given axis. (See Product of Inertia of a Section and Chapter 5.)

Natural Frequency (Eigenvalue)–The frequency at which a linear elastic structure will tend to vibrate once it has been set into motion. A structure can possess many natural frequencies. The lowest of these is called the fundamental natural frequency. Each natural frequency is associated with a mode shape of deformation. Natural frequency can be defined either in terms of cycles per second (hertz) or radians per second. There are 2π radians per cycle.

Neutral Axis–The axis of zero stress in the cross section of a structure. The neutral axis must pass through the centroid of the cross section of homogeneous beams if the axial load is zero so that the beam supports only a bending load.

Node–A point on a structure which does not deflect during vibration in a given mode. Anti-node is a point on a structure where deflection is maximum during vibration in a given mode.

Orthotropic–A term applied to a thin lamina if the material properties of the lamina possess two mutually perpendicular planes of symmetry. Four material constants are required to specify the elastic behavior of an orthotropic lamina. Common examples of orthotropic lamina are sheets of fiber-reinforced plastic or the thin plys of wood that are glued together to form plywood.

Pinned Boundary–A boundary condition such that the structure is free to rotate but not displace along a given boundary.

Plate—A thin two-dimensional elastic structure which is composed of material in the vicinity of a flat two-dimensional sheet. A plate without bending rigidity is a membrane.

Point Mass (Concentrated Mass)—A point in space having mass, but zero moment of inertia for rotation about its center of mass.

Poisson's Ratio—The ratio of the lateral shrinkage (expansion) to the longitudinal expansion (shrinkage) of a bar of a given material which has been placed under a uniform longitudinal tensile (compressive) load. Poisson's ratio is ordinarily near 0.3 and is dimensionless. Some materials, such as wood, have a directional Poisson's ratio. For most materials, within the limits of elasticity, Poisson's ratio is independent of the sign of the applied stress.

Product of Inertia of a Body—The sum of the products obtained by multiplying each element of mass of a body by the distances from two mutually perpendicular axes. (See Chapter 5.)

Product of Inertia of a Section—The sum of the products obtained by multiplying each element of area of a section by the distances from two mutually perpendicular axes. (See Chapter 5.)

Radius of Gyration of a Body—The square root of the quantity formed by dividing the mass moment of inertia of a body by the mass of the body. (See Chapter 5.)

Radius of Gyration of a Section—The square root of the quantity formed by dividing the area moment of inertia of a section by the area of the section. (See Chapter 5.)

Rotary Inertia—The inertia associated with local rotation of a structure. For example, the rotary inertia of a spinning top maintains its rotation.

Seiching—The system of waves in a harbor which is produced as the harbor responds sympathetically to waves in the open sea (also see Sloshing).

Shear Beam—A beam whose deformation in shear substantially exceeds the flexural deformation.

Shear Coefficient—A dimensionless quantity, dependent on the shape of the cross section of a beam, which is introduced into approximate beam theory to account for the fact that shear stress and shear strain are not uniformly distributed over the cross section. The shear coefficient is generally defined as the ratio of the average shear strain over the beam cross section to the shear strain at the centroid. See Section 8.2.

Shear Modulus—The rate change in the shear stress of a material with a unit shear strain. For most materials the shear modulus is independent of the sign of the applied stress, although some materials, such as wood, may have a directional shear modulus. $G = E/[2(1 + \nu)]$ for an isotropic elastic material, that is, a material whose properties are the same in all directions. (Definitions of symbols are given in Chapter 2.)

Shell—A thin elastic structure whose material is confined to the close vicinity of a curved surface, the middle surface of the shell. A curved plate is a shell. A shell without rigidity in bending is a membrane.

Sliding Boundary—A boundary condition such that a structure is free to displace in a given direction along a boundary but rotation is prevented.

Sloshing—The system of surface waves formed in a liquid-filled tank or basin as the liquid is excited.

Speed of Sound—The speed at which very small pressure fluctuations propagate in a infinite fluid or solid.

Spring Constant (Deflection)—The change in load on a linear elastic structure required to produce a unit increment of deflection.

Spring Constant (Torsion)—The change in moment (torque) on a linear elastic structure required to produce a unit increment in rotation.

String—A massless one-dimensional structure which can only bear tension parallel with its own axis. A string is a massless cable.

Viscosity—The ability of a fluid to resist shearing deformation. The viscosity of a linear (Newtonian) fluid is defined as the ratio between the shear stress applied to a fluid and the shearing strain that results. Kinematic viscosity is defined as viscosity divided by fluid density.

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SYMBOLS

Throughout this book, definitions of symbols are given at the top of each table and in the text. In some cases special symbols have been defined. The symbols listed below have been consistently applied in all cases. These symbols generally follow those used in the literature. One exception is that here I is used to denote all area moments of inertia of sections and J is used to denote all mass moments of inertia of bodies.

A	area ($length^2$)
B	bulk modulus (force/area)
C	center of gravity or centroid, also torsion constant ($length^4$)
E	modulus of elasticity (force/area)
G	shear modulus (force/area)
I	area moment of inertia ($length^4$)
J	mass moment of inertia (mass $\times$ $length^2$)
K	shear coefficient (dimensionless)
L	length
M	mass
P	load (force)
S	tension per unit length of edge (force/length) or length
T	tension (force)
X, Y, Z	mutually orthogonal displacements (length)
c	speed of sound (length/time)
f	frequency (hertz)
g	acceleration of gravity (length/$time^2$) or grams
k	deflection spring constant (force/length)
m	mass per unit length (mass/length)
p	load per unit length (force/length) or pressure (Chapter 14)
x, y, z	mutually orthogonal coordinates (length)
$\tilde{x}$, $\tilde{y}$, $\tilde{z}$	mode shapes associated with the X, Y, and Z displacements, respectively (dimensionless)
α	angle (radians) or dimensionless constant
γ	mass per unit area (mass/$length^2$) or ratio of specific heats (dimensionless)
ϵ	strain (dimensionless)
θ	rotation (radians)
$\tilde{\theta}$	mode shapes associated with θ rotation (dimensionless)
μ	material density (mass/$length^3$)
ν	Poisson's ratio
π	= 3.1415926