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Parametric Optimization and Approximation

Edited by

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Editors

B. Brosowski
Universität Frankfurt
Fachbereich Mathematik
Robert-Mayer-Str. 10
D–6000 Frankfurt (FRG)

F. Deutsch
Pennsylvania State University
Department of Mathematics
215 McAllister Building
University Park, PA 16802 USA

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PREFACE.

This volume contains the proceedings of the International Symposium on "Parametric Optimization and Approximation", held at the Oberwolfach Research Institute, Black Forest, October 16-22, 1983. It includes papers either on a research or of an advanced expository nature. Some of them could not actually be presented during the symposium, and are being included here by invitation. The participants came from Brazil, Bulgaria, CSSR, German Democratic Republic, Great Britain, Israel, Netherlands, South Africa, USA, and West Germany. We take the opportunity to express our thanks to all those who participated in the symposium or contributed to this volume. We also thank the Oberwolfach Mathematical Research Institute for the facilities provided.

August 1984.

Bruno Brosowski, Frankfurt a.M.

Frank Deutsch, University Park, PA

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A RITZ METHOD FOR THE NUMERICAL SOLUTION OF A CLASS OF STATE CONSTRAINED CONTROL APPROXIMATION PROBLEMS

Walter Alt

Mathematisches Institut
Universität Bayreuth
Postfach 3008
D-8580 Bayreuth

Introduction

This paper is concerned with a control approximation problem which occurs in connection with the optimal heating of solids. We consider a one-dimensional homogeneous metal rod which is kept insulated at the left end, and is heated at the right end, where the temperature is regulated by a control function. The problem consists of finding an optimal control such that the deviation of the temperature distribution in the rod at a fixed final time from a desired distribution is minimized; at the same time the temperature has to satisfy certain constraints. We use a Ritz type method to approximate the original problem by a series of discrete convex optimization problems, and we derive error bounds for the extremal values of the discrete problems.

1. The control approximation problem

We consider the following problem:

(P) Minimize $\int_0^1 (y(T,x) - y_T(x))^2 \, dx$

subject to $y \in C([0,T] \times [0,1])$, $u \in L^\infty[0,T]$ and

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = 0, \tag{1.1}$$

$$\frac{\partial y}{\partial x}(\cdot,0) = 0, \tag{1.2}$$

$$\alpha y(\cdot,1) + \frac{\partial y}{\partial x}(\cdot,1) = u, \tag{1.3}$$

$$y(0,\cdot) = 0, \tag{1.4}$$

$$\rho_1 \leqslant u(t) \leqslant \rho_2 \qquad \text{a.e. on } [0,T]. \tag{1.5}$$

$$y(t,1) \leqslant \eta(t) \qquad \forall\, t \in [0,T]. \tag{1.6}$$

Herein T, α, ρ_1, ρ_2 are given real numbers such that $T > 0$, $\alpha > 0$ and $\rho_1 < \rho_2$; y_T and η are fixed functions with $y_T \in L^2[0,1]$, $\eta \in C[0,T]$.

Let $p \in \,]2,\infty]$ and $u \in L^p[0,T]$ be given. In $L^2[0,1]$ the generalized solution $y(u)$ of (1.1) - (1.4) has the series representation

$$y(u)(t) = \sum_{k=1}^{\infty} v_k(1) \int_0^t e^{-\lambda_k(t-\tau)} u(\tau)\, d\tau \; v_k \qquad \forall\, t \in [0,T] \tag{1.7}$$

where the λ_k resp. v_k are the eigenvalues resp. eigenfunctions of the corresponding elliptic eigenvalue problem. The operator S defined by

$$S u := (y(u), y(u)(T)) \qquad \forall\, u \in L^p[0,T] \tag{1.8}$$

is a continuous linear operator from $L^p[0,T]$ into $C([0,T] \times [0,1]) \times L^2[0,1]$ (compare ALT/MACKENROTH [2] and MACKENROTH [5]).

By p_1 resp. p_2 we denote the canonical projection of $C([0,T] \times [0,1]) \times L^2[0,1]$ onto $C([0,T] \times [0,1])$ resp. $L^2[0,1]$. Further we define

$$(1.9) \qquad K := \{z \in C([0,T] \times [0,1]) \mid z(t,1) \leq \eta(t) \ \forall\, t \in [0,T]\}.$$

Then problem (P) can be written in the following form:

(PA) Minimize $\|p_2 S u - y_T\|_2^2$
subject to $u \in L^\infty[0,T]$ and

$$(1.10) \qquad \rho_1 \leq u(t) \leq \rho_2(t) \qquad \text{a.e. on } [0,T],$$

$$(1.11) \qquad p_1 S u \in K.$$

2. The Ritz method

For the numerical solution of problem (P) we present a Ritz type method which defines a series of discrete optimization problems approximating the original problem. The original control space is replaced by a finite dimensional one, and the operator S is approximated by a finite series based on (1.7). To this end let for $i \in \mathbb{N}$ numbers $n_i, k_i, m_i \in \mathbb{N}$ be given and decompositions

$$(2.1) \qquad \begin{aligned} 0 &= t_0^i < t_1^i < \dots < t_{n_i}^i = T, \\ 0 &= s_0^i < s_1^i < \dots < s_{m_i}^i = T. \end{aligned}$$

Let $u_\nu^i : [0,T] \to \mathbb{R}$ be defined by

$$(2.2) \qquad u_\nu^i(t) = \begin{cases} 1 & \text{if } t \in [t_{\nu-1}^i, t_\nu^i] \\ 0 & \text{elsewhere} \end{cases} \qquad (\nu = 1,\dots,n_i)$$

The original control space $U = L^\infty[0,T]$ is replaced by the finite dimensional subspace of piecewise constant functions

$$(2.3) \qquad U_i := \operatorname{span}\{u_1^i,\dots,u_{n_i}^i\}.$$

The operator S is approximated by the finite sum

$$(2.4) \qquad (p_1 S_i u)(t,x) := \sum_{k=1}^{k_i} v_k(1) \int_0^t e^{-\lambda_k (t-\tau)} u(\tau)\, d\tau \; v_k(x).$$

With these notations we can formulate the discrete approximations to (P) resp. (PA) as follows:

(P_i) Minimize $\| p_2 S_i u - y_T \|_2^2$
subject to $u \in U_i$ and

(2.5) $\rho_1 \leq u(t) \leq \rho_2$ a. e. on $[0,T]$,

(2.6) $p_1 S_i u \in K_i$

where the set K_i is defined by

(2.7) $K_i = \{z \in C([0,T] \times [0,1]) \mid z(s_\nu^i, 1) \leq \eta(s_\nu^i),$
$\nu = 0,1,\ldots,m_i\}.$

Problem (P_i) defines a finite dimensional quadratic optimization problem which can be solved by a suitable numerical procedure.

In order to derive convergence results for the extremal values of the problems (P_i) we need the following Slater condition:

(2.8) There is a control $\bar{u} \in L^\infty[0,T]$ with (1.5) and
$(p_1 S \bar{u})(t,1) < \eta(t) \quad \forall\, t \in [0,T].$

Let

(2.9)
$\tau_i .= \max \{t_\nu^i - t_{\nu-1}^i \mid \nu = 1,\ldots,n_i\}$
$\sigma_i .= \max \{s_\nu^i - s_{\nu-1}^i \mid \nu = 1,\ldots,m_i\}.$

In ALT/MACKENROTH [3] we have shown the following result.

Theorem 2.1. Suppose that the Slater condition (2.8) is satisfied and $\lim_{i\to\infty} \tau_i = 0$, $\lim_{i\to\infty} \sigma_i = 0$, $\lim_{i\to\infty} k_i = \infty$. Then there is a number $i_o \in \mathbb{N}$ such that for $i \geq i_o$ (P_i) has an optimal solution and

$$\lim_{i\to\infty} \inf (P_i) = \inf (P).$$

The aim of this paper is to derive in addition error bounds for $|\inf (P_i) - \inf (P)|$. To this end we use methods similiar to those developed in ALT [1].

3. Error bounds for the extremal values

We start by presenting three auxiliary results which we need for our convergence analysis.

Lemma 3.1. Let U, Z be Banach spaces, $K \subset Z$, $A \in \mathcal{L}(U,Z)$. Suppose that for every $i \in \mathbb{N}$ a subset $K_i \subset Z$ and an operator $A_i \in \mathcal{L}(U,Z)$ are given such that the following conditions are satisfied.

(a) There is an $\bar{u} \in U$ with $A\bar{u} \in \operatorname{int} K$.

(b) $\lim_{i\to\infty} A_i u = Au \quad \forall\, u \in U$.

(c) $K \subset K_i$.

Then there is a real number $\eta > 0$, and for any sequence $\{u_i\} \subset U$ with $\lim_{i\to\infty} u_i = \bar{u}$ there is an $i_o \in \mathbb{N}$ with

$$(3.1) \qquad \eta B_Z \in A_i u_i - K_i \qquad \forall\, i \geq i_o.$$

Proof. By (a) there is a $\mu > 0$ with $A\bar{u} - \mu B_Z \subset K$. Let $\{u_i\} \subset U$ be a sequence with $\lim_{i\to\infty} u_i = \bar{u}$. From (b) and the theorem of Banach-Steinhaus we obtain $\lim_{i\to\infty} A_i u_i = A\bar{u}$. Hence, there is an $i_o > 0$ with

$$\|A_i u_i - A\bar{u}\| \leq \frac{\mu}{2} \qquad \forall\, i \geq i_o.$$

This implies

$$A_i u_i - \eta B_Z \subset K \subset K_i$$

for $\eta := \frac{\mu}{2}$. □

The proof of the following lemma is based on the proof of theorem 2 in ROBINSON [6].

Lemma 3.2. Let U, Z be Banach spaces, $C \subset U$, $K \subset Z$ closed convex sets, and $A \in \mathcal{L}(U,Z)$. Suppose that for every $i \in \mathbb{N}$ closed convex sets $C_i \subset U$, $K_i \subset Z$ and an operator $A_i \in \mathcal{L}(U,Z)$

are given such that assumptions (b), (c) of lemma 3.1 and the following conditions are satisfied:

(a') There is an $\bar{u} \in C$ with $A\bar{u} \in \operatorname{int} K$.

(d) For every $u \in C$ there is a sequence $\{u_i\} \subset U$ with $u_i \in C_i$ and $\lim_{i\to\infty} u_i = u$.

(e) $\|u\| \leqslant r \quad \forall\, u \in C$ and $\forall\, u \in C_i$ for some constant r.

Define $D := \{u \in U \mid Au \in K\}$ and

$D_i := \{u \in U \mid A_i u \in K_i\}$.

Then there is real number $\eta > 0$ such that the following holds: If $u_o \in C \cap D$ and $\{u_i\} \subset U$ is a sequence with $u_i \in C_i \;\forall\, i \in \mathbb{N}$ and $\lim_{i\to\infty} u_i = u_o$ then there is a $i_o \in \mathbb{N}$ and a sequence $\{v_i\}$ such that for all $i \geqslant i_o$

(3.2) $\quad v_i \in C_i \cap D_i$

(3.3) $\quad \|v_i - u_o\| \leqslant \|u_i - u_o\| + \frac{2r}{\eta} \|A_i u_i - A u_o\|$.

Proof. Assumption (d) implies the existence of a sequence $\{\bar{u}_i\}$ with $\bar{u}_i \in C_i$ and $\lim_{i\to\infty} \bar{u}_i = \bar{u}$. Hence by lemma 3.1 there are $\eta > 0$ and $i_o \in \mathbb{N}$ with

$$\eta B_Z \subset A_i\bar{u}_i - K_i \qquad \forall\, i \geqslant i_o.$$

In particular we have $A_i\bar{u}_i \in \operatorname{int} K \;\forall\, i \geqslant i_o$.

Now let $u_o \in C \cap D$, a sequence $\{u_i\}$ with $u_i \in C_i$, $\lim_{i\to\infty} u_i = u_o$ and $i \geqslant i_o$ be given. If $A_i u_i \in K_i$ we choose $v_i := u_i$; this implies $v_i \in C_i \cap D_i$ and $\|v_i - u_o\| = \|u_i - u_o\|$. Hence (3.2) and (3.3) are satisfied. If $A_i u_i \notin K_i$ we set $d_i := d[A_i u_i, K_i]$. Then for any $\delta > 0$ there is a $k_\delta \in K_i$ such that $z_\delta := A_i u_i - k_\delta$ satisfies the inequalities

$$0 < \|z_\delta\| < d_i + \delta.$$

For $\varepsilon \in]0,\eta[$ we define

$$z_\varepsilon := -(\eta-\varepsilon)\|z_\delta\|^{-1} z_\delta.$$

It follows $\|z_\varepsilon\| \leqslant \eta - \varepsilon < \eta$ and therefore $z_\varepsilon \in \eta B_Z$. Hence there exists $k_\varepsilon \in K_i$ with $z_\varepsilon = A_i\bar{u}_i - k_\varepsilon$. With $\lambda := [1 + (\eta-\varepsilon)\|z_\delta\|^{-1}]^{-1}$ we obtain $0 < \lambda < 1$ and

$$(1-\lambda)z_\delta + \lambda z_\varepsilon = \Theta_Z =$$
$$(1-\lambda)(A_iu_i - k_\delta) + \lambda(A_i\bar{u}_i - k_\varepsilon) =$$
$$A_i((1-\lambda)u_i + \lambda\bar{u}_i) - ((1-\lambda)k_\delta + \lambda k_\varepsilon)$$

For $v_i := (1-\lambda)u_i + \lambda\bar{u}_i$, $k_i := (1-\lambda)k_\delta + \lambda k_\delta$ this implies $v_i \in C_i$, $A_iv_i = k_i \in K_i$ and therefore $v_i \in C_i \cap D_i$. Further we have

$$\|v_i - u_o\| \leqslant \|u_i - u_o\| + \|u_i - v_i\|$$

and

$$\|u_i - v_i\| = \|u_i - (1-\lambda)u_i - \lambda\bar{u}_i\| = \lambda\|u_i - \bar{u}_i\|.$$

From $\lambda \leqslant (\eta-\varepsilon)^{-1}\|z_\delta\|$, $\|z_\delta\| < d_i + \delta$ and $\|u_i - \bar{u}_i\| \leqslant 2r$ we get

$$\lambda\|u_i - \bar{u}_i\| \leqslant 2r(\eta-\varepsilon)^{-1}(d_i + \delta).$$

From this we finally obtain

$$\|v_i - u_o\| \leqslant \|u_i - u_o\| + \frac{2r}{\eta-\varepsilon}(d[A_iu_i,K_i] + \delta).$$

By the fact that $A\,u_o \in K \subset K_i$ the proof is completed by letting δ and ε approach zero. □

Lemma 3.3. Let U, Z be Banach spaces, $C \subset U$, $K \subset Z$ closed convex sets and $A \in \mathcal{L}(U,Z)$. Define $D := \{u \in U \mid A\,u \in K\}$. Suppose that $u_o \in C \cap D$ and that there is an $\bar{u} \in C$ with $A\,\bar{u} \in \operatorname{int} K$. Then there is a real number $\eta > 0$ such that for any $u \in U$ there exists $\tilde{u} \in C \cap D$ with

$$\|u-\tilde{u}\| \leqslant \frac{2}{\eta}(\|\bar{u} - u_o\| + \|u - u_o\|)\, d[Au,K]. \tag{3.5}$$

Proof. Define the multivalued function $F : U \to Z$ by

$$F(u) = \begin{cases} Au - K, & u \in C \\ \emptyset\;\;, & u \notin C. \end{cases}$$

Then F is a closed convex function and by the assumptions of the theorem there exists $\eta > 0$ with

$$\eta B_Z \subset F(u_o + \|\bar{u} - u_o\| B_U).$$

The assertion of the theorem is therefore a special case of theorem 2 in ROBINSON [6]. □

In order to state our main result we introduce some notations. The cost functional of problem (P) resp. (PA) is denoted by f, i. e.

$$f(u) := \|p_2 S u - y_T\|_2^2 .$$

The cost functional of problem (P_i) is denoted by f_i, i. e.

$$f_i(u) := \|p_2 S_i u - y_T\|_2^2 .$$

Further, if $p \in]2,\infty]$ is given and q is defined by $\frac{1}{p} + \frac{1}{q} = 1$ for $p < \infty$ and $q = 1$ for $p = \infty$, we can define a continuous function g by

$$(3.6) \qquad g(s) = \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{q}} (1 - e^{\lambda_k q s}) \qquad \forall\, s > 0.$$

We can now formulate our main result.

Theorem 3.4. Let u* be an optimal solution of (P), and suppose that the Slater-condition (2.8) is satisfied. Let $p \in]2,\infty]$ and a sequence $\{u_i\}$ be given with $\rho_1 \leqslant u_i(t) \leqslant \rho_2$ a. e. on $[0,T]$ and $\lim_{i\to\infty} \|u_i - u^*\|_p = 0$. Then there are constants c_1, c_2, c_3, c_4 and an $i_o \in \mathbb{N}$ such that for $i \geqslant i_o$ the following holds.

(3.7) (P_i) has an optimal solution u_i^*.

(3.8) $f_i(u_i^*) - f(u^*) \leqslant c_1 \|S_i - S\| + c_2 \|u_i - u^*\|_p$

(3.9) $f(u^*) - f_i(u_i^*) \leqslant c_3 \|S_i - S\| + c_4\, g(\sigma_i)$
where σ_i is defined by (2.9).

Proof. Assertion (3.7) was shown in ALT/MACKENROTH [4]. To proof (3.8) and (3.9) we define $U = L^p[0,T]$, $C = \{u \in U \mid u \text{ satisfies } (1.5)\}$, $C_i = C \cap U_i$ and $r = \max\{|\rho_1|, |\rho_2|\}$. Further we will use the fact that there are constants $\tilde{c}_1, \tilde{c}_2$ with