Theory of Set Differential Equations in Metric Spaces

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Preface

The study of analysis in metric spaces has gained importance in recent times. It is realized that many results of differential calculus and set valued analysis, including the inverse function theorem do not really rely upon the linear structure and therefore can be adapted to the nonlinear case of metric spaces and exploited. Moreover, the concept of the differential equation governing evolution in metric spaces has been suitably formulated.

Multivalued differential equations (now known as set differential equations (SDEs)) generated by multivalued differential inclusions have been introduced in a semi-linear metric space, consisting of all nonempty, compact, convex subsets of an initial finite or infinite dimensional space. The basic existence and uniqueness results of such SDEs have been investigated and their solutions have compact, convex values. Also, these generated SDEs have been employed as a tool to prove the existence of solutions, in a unified way, of multivalued differential inclusions. The multifunctions involved in this set up are compact, but not necessarily convex, subsets of the base space utilized.

Because of the fact that fuzzy set theory and its applications have been extensively investigated, due to the increase of industrial interest in fuzzy control, the initiation of the theory of fuzzy differential equations (FDEs) in an appropriate metric space has recently been accomplished. In view of the inherent disadvantage resulting from the fuzzification of the derivative employed in the original formulation of FDEs, an alternative formulation based upon a family of multivalued differential inclusions derived from the fuzzy maps involved in the FDEs, is recently suggested to reflect the rich behaviour of the corresponding ordinary differential equation before fuzzification.

The investigation of the theory of SDEs as an independent discipline, has certain advantages. For example, when the set is a single valued mapping, it is clear that the Hukuhara derivative and the integral utilized in formulating the SDEs reduce to the ordinary vector derivative and the integral, and therefore the results obtained in the framework of SDEs become the corresponding results of ordinary differential systems if the base space is $\mathbb{R}^n$. On the other hand, if the base space is a Banach space, we get from the corresponding SDE's the differential equations in a Banach space. Moreover, one has only a semilinear metric space to work with in the SDE set up, compared to the complete normed linear space that one employs in the usual study of an ordinary differential system. As indicated earlier,the SDEs that are generated by multivalued differential inclusions when the needed convexity is missing, form a natural vehicle for proving the existence results for multivalued differential inclusions. Also, one can utilize SDEs profitably to investigate FDEs. Consequently, the study of the theory of SDEs has recently been growing very rapidly and is still in the initial stages. Nonetheless, there exists sufficient literature to warrant assembling the existing fundamental results in a unified way to understand and appreciate the intricacies and advantages involved, so as to pave the way for further advancement of this important branch of differential equations as an independent subject area. It is with this spirit we see the importance of the present monograph. As a result, we provide a systematic account of recent development, describe the current state of the useful theory, show the essential unity achieved and initiate several new extensions to other types of SDEs.

In Chapter 1, we assemble the preliminary material providing the necessary tools including the calculus for set valued maps relevant to the later development. Chapter 2 is devoted to the investigation of the fundamental theory of SDEs such as various comparison principles, existence and uniqueness, continuous dependence, existence of extremal solutions suitably introducing a partial order in the metric space, monotone iterative technique using lower and upper solutions and global existence under the continuity assumption for SDEs. We also discuss, utilizing the method of nonsmooth analysis, existence and flow invariance results without any continuity assumption, in terms of Euler solutions. Finally, we consider the case of upper semicontinuity in the framework of Caratheodory and prove an existence result in a general set up.

In Chapter 3, we extend Lyapunov stability theory to SDEs, employing Lyapunov-like functions, proving first suitable comparison results in terms of such functions. The stability and boundedness criteria are obtained by choosing appropriate initial values in terms of the Hukuhara difference to eliminate the undesirable part of the solutions of SDEs, so that the rich behaviour of the corresponding ODEs, from which SDEs are generated, is preserved. The methods of vector Lyapunov-like functions and the perturbing Lyapunov-like functions are discussed in detail. Also, employing lower semicontinuous Lyapunov-like functions and utilizing nonsmooth analysis, stability results are described under weaker assumptions.

Chapter 4 deals with the interconnection between SDEs and fuzzy differential equations(FDEs). For this purpose, necessary tools are provided for formulating FDEs, and basic results are proved, including the stability theory of Lyapunov. Then the interconnection between FDEs and SDEs is explored via a sequence of multivalued differential inclusions, suitably generating SDEs as described earlier. The impulsive effects are then incorporated in FDEs and then it is shown how impulses can help to improve the qualitative behaviour of solutions of FDEs. Hybrid fuzzy differential equations are introduced and their stability properties are discussed. Another concept of differential equations in metric spaces is considered which can be applied to the study of FDEs.

Chapter 5 is devoted to initiate several topics in the setup of SDEs such as impulsive SDEs, SDEs with time delay, set difference equations, and SDEs involving causal maps, which cover several types of SDEs including integrodifferential equations. Some important basic results are provided for each type of SDEs. We then introduce Lyapunov-like functions whose values are in some metric space, prove suitable comparison results and study stability theory in this general set up. This study includes the methods of single, vector, matrix and cone-valued Lyapunov-like functions by an appropriate choice of the metric space. Since the basic space utilized to define the metric space $(K_c(\mathbb{R}^n), D)$ is restricted, for convenience of understanding, to $\mathbb{R}^n$, we indicate how one can extend most of the results described when we choose a Banach space E instead of $\mathbb{R}^n$, so that we have the corresponding metric space $(K_c(E), D)$ to work with. Finally, notes and comments are provided for each chapter.

Some of the important features of the monograph as follows:

1. It is the first book that attempts to describe the theory of set differential equations as an independent discipline.;

2. It incorporates, the recent general theory of set differential equations, discusses the interconnections between set differential equations and fuzzy differential equations and uses both smooth and nonsmooth analysis for investigation.

3. It exhibits several new areas of study by providing the initial apparatus for further advancement.

4. It is a timely introduction to a subject that follows the present trend of studying analysis and differential equations in metric spaces.

This monograph will be very useful to those experts and their doctoral students who work in Nonlinear Analysis, in general. It will also be a good reference book to Engineering and Computer Scientists since it also covers fuzzy dynamics as a subset.

We are immensely thankful to Professors Alex Tolstanogov, Donta Satyanarayana and S. Leela for their valuable suggestions and help and to Mrs. Donn Miller-Kermani for the excellent CRC typing of the manuscript. We are pleased, indeed, to offer our heartfelt thanks to Ms. Janie Wardle for her interest and cooperation in our projects.

Contents

Chapter 1

Preliminaries

1.1 Introduction

Recently the study of set differential equations (SDEs) was initiated in a metric space and some basic results of interest were obtained. The investigation of set differential equations as an independent subject has some advantages. For example, when the set is a single valued mapping, it is easy to see that the Hukuhara derivative, and the integral utilized in formulating the SDEs reduce to the ordinary vector derivative and the integral and therefore, the results obtained in this new framework become the corresponding results in ordinary differential systems. Also, we have only a semilinear complete metric space to work with in the present setup, compared to the normed linear space that one employs in the usual study of ordinary differential systems.

Furthermore, SDEs that are generated by multivalued differential inclusions, when the multivalued functions involved do not possess convex values, can be used as a tool for studying multivalued differential inclusions. Moreover, one can utilize SDEs indirectly to investigate profitably fuzzy differential equations, since the original formulation of fuzzy differential equations suffers from grave disadvantage and does not reflect the rich behavior of corresponding differential equations without fuzziness. This is due to the fact that the diameter of any solution of a fuzzy differential equation increases as time increases because of the necessity of the fuzzification of the derivative involved.

In order to formulate the set differential equations in a metric space, we need some background material, since the metric space involved consists of all nonempty compact, convex sets in finite or infinite dimensional space. In Section 1.2, we define the necessary ingredients of such sets restricting ourselves to the Euclidean n-space $\mathbb{R}^n$. Since the difference of any two sets in $K_c(\mathbb{R}^n)$ (set of all nonempty, compact, convex sets in $\mathbb{R}^n$) is not defined in general, conditions for the existence of the difference are provided in this section. Section 1.3 introduces the Hausdorff metric $D[\cdot,\cdot]$ for $K_c(\mathbb{R}^n)$ and lists its properties. Support functions are defined in Section 1.4, where they are utilized to create a mapping that makes it possible to embed the metric space $(K_c(\mathbb{R}^n), D)$ into a complete cone in a specified Banach space.

In Section 1.5, the continuity and measurability properties of mappings into the metric space are dealt with. Section 1.6 investigates the concept of differentiation of such mappings and its behavior. In Section 1.7, we consider the theory of integration of these mappings and the needed properties. Section 1.8 summarizes the corresponding situation when the elements of the metric space considered are from a Banach space. Notes and comments are listed in Section 1.9.

1.2 Compact Convex Subsets of $\mathbb{R}^n$

We shall consider the following three spaces of nonempty subsets of $\mathbb{R}^n$, namely,

(i) $K_c(\mathbb{R}^n)$ consisting of all nonempty compact convex subsets of $\mathbb{R}^n$;

(ii) $K(\mathbb{R}^n)$ consisting of all nonempty compact subsets of $\mathbb{R}^n$;

(iii) $C(\mathbb{R}^n)$ consisting of all nonempty closed subsets of $\mathbb{R}^n$.

Recall that a nonempty subset A of $\mathbb{R}^n$ is convex if for all $a_1,\ a_2 \in A$ and all $\lambda \in [0,1]$, the point

$$a = \lambda a_1 + (1-\lambda)a_2 \tag{1.2.1}$$

belongs to A. For any nonempty subset A of $\mathbb{R}^n$, we denote by co$A$ its *convex hull*, that is the totality of points a of the form (1.2.1) or, equivalently, the smallest convex subset containing A. Clearly

$$A \subseteq co\ A = co(co\ A) \tag{1.2.2}$$

with $A = coA$ if A is convex. Moreover, co$A$ is closed (compact) if A is closed (compact).

Let A and B be two nonempty subsets of $\mathbb{R}^n$ and let $\lambda \in I\!R$. We define the following Minkowski addition and scalar multiplication by

$$A + B = \{a + b : a \in A,\ b \in B\} \tag{1.2.3}$$

and

$$\lambda A = \{\lambda a :\ a \in A\}. \tag{1.2.4}$$

Then we have the following proposition.

Proposition 1.2.1 *The spaces $C(\mathbb{R}^n)$, $K(\mathbb{R}^n)$ and $K_c(\mathbb{R}^n)$ are closed under the operations of addition and scalar multiplication. In fact, the following properties hold:*

(i) $A+\theta = \theta+A = A$, $\theta \in \mathbb{R}^n$, *is the zero element of* $\mathbb{R}^n$, *treated as a singleton.*

(ii) $(A+B)+C = A+(B+C)$

(iii) $A + B = B + A$

(iv) $A + C = B + C$ *implies* $A = B$

(v) $1 \cdot A = A$

(vi) $\lambda(A + B) = \lambda A + \lambda B$

(vii) $(\lambda + \mu)A = \lambda A + \mu A$

where $A, B, C \in K_c(\mathbb{R}^n)$, λ, $\mu \in I\!R_+$.

Proof We only give the proof of (iv), the rest being simple to prove.

Let A, B, $C \in K_c(\mathbb{R}^n)$. We show that $A \neq B$ implies $A + C \neq B + C$. Suppose, for example, that there exists a point $a \in A$ which does not belong to B. Through a pass hyperplanes which are disjoint from B. Let one of these hyperplanes be P. Let P' be the support hyperplane of C, which is parallel to P and such that, if we move P' parallel to itself onto P, C moves on a compact convex set which is located on the same side of P as B. If c is a point of $C \cap P'$, then $a + c \notin B + C$. Hence the proof.

In general, $A+(-A) \neq \{\theta\}$. This fact is illustrated by the following example.

Example 1.2.1 *Let* $A = [0, 1]$ *so that* $(-1)A = [-1, 0]$, *and therefore*

$$A + (-1)A = [0, 1] + [-1, 0] = [-1, 1].$$

Thus, adding (-1) *times a set does not constitute a natural operation of subtraction.*

This leads us to the following definition.

Definition 1.2.1 *For a fixed* A *and* B *in* $K_c(\mathbb{R}^n)$ *if there exists an element* $C \in K_c(\mathbb{R}^n)$ *such that* $A = B + C$ *then we say that the Hukuhara Difference of* A *and* B *exists and is denoted by* $A - B$.

When the Hukuhara difference exists it is unique. This follows from (iv) of Proposition 1.2.1.

The following example explains the above definition.

Example 1.2.2 *From Example 1.2.1, we get*

$$[-1, 1] - [-1, 0] = [0, 1] \quad and \quad [-1, 1] - [0, 1] = [-1, 0].$$

Note that the Hukuhara difference $A - B$ is different from the set

$$A + (-B) = \{a + (-b) : \ a \in A, \ b \in B\}.$$

The next proposition provides the necessary and sufficient condition for the existence of the Hukuhara difference $A - B$.

Proposition 1.2.2 *Let $A, B \in K_c(\mathbb{R}^n)$. For the difference $A - B$ to exist, it is necessary and sufficient to have the following condition. If $a \in \partial A$, there exists at least a point c such that*

$$a \in B + c \subset A. \tag{1.2.5}$$

Proof Necessity: Suppose the difference $A - B$ exists. Let $C = A - B$. Then, $A = B + C$. If $a \in \partial A$, $a \in B + C$, that is, $a = b + c$ where $b \in B$ and $c \in C$. Also, if $z \in B$, then $z + c \in A$ and therefore (1.2.5) is satisfied.

Sufficiency: Suppose (1.2.5) holds. Consider the set $C = \{x : B + x \subseteq A\}$. Clearly C is compact and we have $B + C \subseteq A$. Now, if d and $d' \in C$, then we have $B + d \subseteq A$ and $B + d' \subseteq A$, from which we obtain

$$(1 - \lambda)(B + d) + \lambda(B + d') \subset A, \text{ for } 0 \leq \lambda \leq 1. \tag{1.2.6}$$

We can write the L.H.S of (1.2.6) as $B + z$ with $z = (1 - \lambda)d + \lambda d'$. Hence $z \in C$ and C is convex.

Let $u \in A$. A straight line through u meets ∂A at two points a and a'. By hypothesis there exist elements d and d' in C such that $a \in B + d$, and $a' \in B + d'$. We can write $u = (1 - \lambda)a + \lambda a'$ with $0 < \lambda < 1$. Then $u \in B + x$, where $x = (1 - \lambda)d + \lambda d' \in C$. Hence $A \subseteq B + C$. Thus $A = B + C$ and the proof is complete.

We note that a necessary condition for the Hukuhara difference $A - B$ to exist is that some translate of B is a subset of A. However, in general, the Hukuhara difference need not exist as is seen from the following example.

Example 1.2.3 $\{0\} - [0, 1]$ *does not exist, since no translate of [0,1] can ever belong to the singleton set* $\{0\}$.

1.3 The Hausdorff Metric

Let x be a point in $\mathbb{R}^n$ and A a nonempty subset of $\mathbb{R}^n$. The distance $d(x, A)$ from x to A is defined by

$$d(x, A) = \inf\{\|x - a\| : a \in A\}. \tag{1.3.1}$$

Thus $d(x, A) = d(x, \bar{A}) \geq 0$ and $d(x, A) = 0$ if and only if $x \in \bar{A}$, the closure of $A \subseteq \mathbb{R}^n$.

We shall call the subset

$$S_\epsilon(A) = \{x \in \mathbb{R}^n : d(x, A) < \epsilon\} \tag{1.3.2}$$

an $\epsilon-$neighborhood of A. Its closure is the subset

$$\bar{S}_\epsilon(A) = \{x \in \mathbb{R}^n : d(x, A) \leq \epsilon\}. \tag{1.3.3}$$

In particular, we shall denote by

$$\bar{S}_1^n = \bar{S}_1(\theta), \tag{1.3.4}$$

which is obviously a compact subset of $\mathbb{R}^n$. Note also that

$$\bar{S}_\epsilon(A) = A + \epsilon\bar{S}_1^n, \tag{1.3.5}$$

for any $\epsilon > 0$ and any nonempty subset A of $\mathbb{R}^n$. We shall for convenience sometimes write $S(A, \epsilon)$ and $\bar{S}_\epsilon(A)$.

Now, let A and B be nonempty subsets of $\mathbb{R}^n$. We define the Hausdorff separation of B from A by

$$d_H(B, A) = \sup\{d(b, A) : b \in B\} \tag{1.3.6}$$

or, equivalently

$$d_H(B, A) = \inf\{\epsilon > 0 : B \subseteq A + \epsilon\bar{S}_1^n\}.$$

We have $d_H(B, A) \geq 0$ with $d_H(B, A) = 0$ if and only if $B \subseteq \bar{A}$. Also, the triangle inequality

$$d_H(B, A) \leq d_H(B, C) + d_H(C, A),$$

holds for all nonempty subsets A, B and C of $\mathbb{R}^n$. In general, however

$$d_H(A, B) \neq d_H(B, A).$$

We define the *Hausdorff* distance between nonempty subsets A and B of $\mathbb{R}^n$ by

$$D(A, B) = \max\{d_H(A, B), d_H(B, A)\}, \tag{1.3.7}$$

which is symmetric in A and B. Consequently,

$$\begin{aligned} &(a) \quad D(A, B) \geq 0 \text{ with } D(A, B) = 0 \text{ if and only if } \bar{A} = \bar{B}; \\ &(b) \quad D(A, B) = D(B, A); \\ &(c) \quad D(A, B) \leq D(A, C) + D(C, B), \end{aligned} \tag{1.3.8}$$

for any nonempty subsets A, B and C of $\mathbb{R}^n$.

If we restrict our attention to nonempty closed subsets of $\mathbb{R}^n$, we find that the Hausdorff distance (1.3.7) is a metric known as the *Hausdorff metric.* Thus $(C(\mathbb{R}^n), D)$ is a metric space.

In fact, we have

Proposition 1.3.1 *$(C(\mathbb{R}^n), D)$ is a complete separable metric space in which $K(\mathbb{R}^n)$ and $K_c(\mathbb{R}^n)$ are closed subsets. Hence, $(K(\mathbb{R}^n), D)$ and $(K_c(\mathbb{R}^n), D)$ are also complete separable metric spaces.*

The following properties of the Hausdorff metric will be useful later.

We start by stating a proposition dealing with the invariance of the Hausdorff metric.

Proposition 1.3.2 *If A, $B \in K_c(\mathbb{R}^n)$ and $C \in K(\mathbb{R}^n)$ then,*

$$D(A + C, B + C) = D(A, B). \tag{1.3.9}$$

We need the following result which deals with the law of cancellation to proceed further.

Lemma 1.3.1 *Let A, $B \in K_c(\mathbb{R}^n)$ and $C \in K(\mathbb{R}^n)$ and $A + C \subseteq B + C$, then $A \subseteq B$.*

Proof Let a be any element of A. We need to show that $a \in B$. Given any $c_1 \in C$, we have $a + c_1 \in B + C$, that is, there exist $b_1 \in B$ and $c_2 \in C$ with $a+c_1 = b_1+c_2$. For the same reason, $b_2 \in B$ and $c_3 \in C$ with $a+c_2 = b_2+c_3$ must exist. Repeat the procedure indefinitely and sum the first n of the equations obtained. We get

$$na + \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} b_i + \sum_{i=2}^{n+1} c_i$$

which implies

$$na + c_1 = \sum_{i=1}^{n} b_i + c_{n+1}.$$

Then,

$$a = \frac{1}{n} \sum_{i=1}^{n} b_i + \frac{c_{n+1}}{n} - \frac{c_1}{n}.$$

Set $x_n = \frac{1}{n} \sum_{i=1}^{n} b_i$. Thus

$$a = x_n + \frac{c_{n+1}}{n} - \frac{c_1}{n}.$$

We observe that $x_n \in B$ for all n, because B is convex and $\frac{c_{n+1}}{n} - \frac{c_1}{n} \to 0$ as C is compact. Thus x_n converges to a. But since B is compact, $a \in B$. Thus, if $A + C = B + C$ then $A = B$. This completes the proof of the lemma.

Proof of Proposition 1.3.2. Let $\lambda \geq 0$ and S denote the closed unit sphere of the space. Consider the following inequalities

(1) $A + \lambda S \supset B$,

(2) $B + \lambda S \supset A$,

(3) $A + C + \lambda S \supset B + C$,

(4) $B + C + \lambda S \supset A + C$.

Put $d_1 = D(A, B)$ and $d_2 = D(A + C, B + C)$. Then d_1 is the infimum of the positive numbers λ for which (1) and (2) hold. Similarly, d_2 is the infimum of the positive numbers λ for which (3) and (4) hold. Since (3) and (4) follow from (1) and (2) respectively, by adding C, we have $d_1 \geq d_2$. Conversely, since by Lemma 1.3.1, canceling C is allowed in (3) and (4), we obtain $d_1 \leq d_2$, which proves the proposition.

Proposition 1.3.3 *If $A, B \in K(\mathbb{R}^n)$*

$$D(co\ A, co\ B) \leq D(A, B). \tag{1.3.10}$$

If $A,\ A',\ B,\ B' \in K_c(\mathbb{R}^n)$ then

$$D(tA, tB) = tD(A, B) \text{ for all } t \geq 0, \tag{1.3.11}$$

$$D(A + A', B + B') \leq D(A, B) + D(A', B'), \tag{1.3.12}$$

Further,

$$D(A - A', B - B') \leq D(A, B) + D(A', B'), \tag{1.3.13}$$

provided the differences $A - A'$ and $B - B'$ exist. Moreover with $\beta = \max\{\lambda,\ \mu\}$, we have

$$D(\lambda A, \mu B) \leq \beta D(A, B) + |\lambda - \mu|[D(A, \theta) + D(B, \theta)] \tag{1.3.14}$$

and

$$D(\lambda A, \lambda B) = \lambda D(A - B, \theta), \text{ if } A - B \text{ exists.} \tag{1.3.15}$$

Proof Since (1.3.10) is obvious, we begin with the proof of (1.3.11). For all $a \in A$ and $u \in A'$, compactness of B and B' ensures that there exist $b(a) \in B$ and $v(u) \in B'$ such that

$$\inf_{b \in B} |a - b| = |a - b(a)|; \quad \inf_{v \in B'} |u - v| = |u - v(u)|. \tag{1.3.16}$$

From the relation

$$|a + u - b(a) - v(u)| \leq |a - b(a)| + |u - v(u)|$$

and (1.3.16), it follows that

$$\sup_{a \in A,\, u \in A'} \inf_{b \in B,\, v \in B'} |a + u - b - v| \leq \sup_{a \in A} \inf_{b \in B} |a - b| + \sup_{u \in A'} \inf_{v \in B'} |u - v|.$$

From the above and the analogous inequality obtained by interchanging the roles of A with B and A' with B', we obtain (1.3.11).

We now prove (1.3.13).

Using Proposition 1.3.2, we find that

$$\begin{aligned} D(A - A', B - B') &= D(A - A' + A' + B',\ B - B' + B' + A') \\ &= D(A + B',\ B + A') \\ &\leq D(A, B) + D(A', B'), \end{aligned}$$

which follows from (1.3.11).

To prove (1.3.14), consider, for $\lambda - \mu \geq 0$,

$$D(\lambda A, \mu B) \leq \mu D(A, B) + (\lambda - \mu) D(A, \theta),$$

and if $\lambda - \mu \le 0$, then

$$D(\lambda A, \mu B) \le \lambda D(A, B) + (\mu - \lambda) D(B, \theta).$$

The relations above put together prove (1.3.14).

The proof of (1.3.15) follows from Proposition 1.3.2.

Next, we define the magnitude of a nonempty subset of A of $\mathbb{R}^n$ by

$$\|A\| = \sup\{\|a\| : a \in A\}, \tag{1.3.17}$$

or equivalently,

$$\|A\| = D(\theta, A). \tag{1.3.18}$$

Here, $\|A\|$ is finite, and the supremum in (1.3.17) is attained when $A \in K(\mathbb{R}^n)$.

From (1.3.10) it obviously follows that

$$\|tA\| = t\|A\|, \quad \text{for all} \quad t \ge 0. \tag{1.3.19}$$

Moreover, (1.3.8) and (1.3.18) yield

$$|\|A\| - \|B\|| \le D(A, B), \tag{1.3.20}$$

for all $A, B \in K(\mathbb{R}^n)$.

We say that a subset $\mathcal{U} \in K(\mathbb{R}^n)$ or $K_c(\mathbb{R}^n)$ is uniformly bounded if there exists a finite constant $c(\mathcal{U})$ such that

$$\|A\| \le c(\mathcal{U}), \quad \text{for all} \quad A \in \mathcal{U}. \tag{1.3.21}$$

We then have the following simple characterization of compactness.

Proposition 1.3.4 *A nonempty subset $\mathcal{A}$ of the metric space $(K(\mathbb{R}^n), D)$ or $(K_c(\mathbb{R}^n), D)$, is compact if and only if it is closed and uniformly bounded.*

Set inclusion induces partial ordering on $K(\mathbb{R}^n)$. Write $A \le B$ if and only if $A \subseteq B$, where $A, B \in K(\mathbb{R}^n)$. Then

$$\mathcal{L}(B) = \{A \in K(\mathbb{R}^n) : B \le A\}, \quad \mathcal{U}(B) = \{A \in K(\mathbb{R}^n) : A \le B\}, \tag{1.3.22}$$

are closed subsets of $K(\mathbb{R}^n)$ for any $B \in K(\mathbb{R}^n)$. In fact, from Proposition 1.3.4, $\mathcal{U}(B)$ is compact subset of $K(\mathbb{R}^n)$.

Proposition 1.3.5 *$\mathcal{U}(B)$ is a compact subset of $K(\mathbb{R}^n)$.*

This assertion remains true with $K_c(\mathbb{R}^n)$ replacing $K(\mathbb{R}^n)$ everywhere.

Sequences of nested subsets in $(K(\mathbb{R}^n), D)$ have the following useful intersection and convergence properties.

Proposition 1.3.6 *Let $\{A_j\} \subset K(\mathbb{R}^n)$ satisfy*

$$\cdots \subseteq A_j \subseteq \cdots A_2 \subseteq A_1.$$

Then $A = \cap_{j=1}^{\infty} A_j \in K(\mathbb{R}^n)$ and

$$D(A_n, A) \to 0 \text{ as } n \to \infty. \tag{1.3.23}$$

On the other hand, if $A_1 \subseteq A_2 \subseteq \cdots A_j \subseteq \cdots$ and $A = \cup_{j=1}^{\infty} A_j \in K(\mathbb{R}^n)$, then (1.3.23) holds.