Progress in Mathematics

Fuensanta Andreu-Vaillo
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Parabolic Quasilinear Equations Minimizing Linear Growth Functionals

INSTITUT D'ESTUDIS CATALANS MCMVII

Ferran Sunyer i Balaguer
Award winning monograph

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Ferran Sunyer i Balaguer (1912–1967) was a self-taught Catalan mathematician who, in spite of a serious physical disability, was very active in research in classical mathematical analysis, an area in which he acquired international recognition. His heirs created the Fundació Ferran Sunyer i Balaguer inside the Institut d'Estudis Catalans to honor the memory of Ferran Sunyer i Balaguer and to promote mathematical research.

Each year, the Fundació Ferran Sunyer i Balaguer and the Institut d'Estudis Catalans award an international research prize for a mathematical monograph of expository nature. The prize-winning monographs are published in this series. Details about the prize and the Fundació Ferran Sunyer i Balaguer can be found at

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This book has been awarded the Ferran Sunyer i Balaguer 2003 prize.

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Preface

Our goal in this monograph is to present general existence and uniqueness results for quasilinear parabolic equations whose operator is, in divergence form, the subdifferential of a Lagrangian which is convex in $|\nabla u|$ and has linear growth as $|\nabla u| \to \infty$. We devote particular attention to the case of the minimizing *total variation flow* for which we study the Neumann, Dirichlet and Cauchy problem in $\mathbb{R}^N$ together with the main qualitative properties of its evolution. This kind of problem appears in different contexts: image processing, faceted crystal growth, continuum mechanics, etc. Motivated by the use of the *total variation model in image restoration*, we started our study of the *minimizing total variation* (TV) *flow* in collaboration with C. Ballester, by studying the corresponding Neumann and Dirichlet problems [13], [14]. Later, in a joint paper with J. I. Diaz [15] we studied the asymptotic behaviour of the solutions of these problems. This study was continued in [34] where some extinction profiles were identified. In particular, this provided some explicit solutions of the denoising problem in image processing. The techniques developed for the total variation flow were extended to cover the case of general convex Lagrangians with linear growth rate in the modulus of the gradient, providing a general existence and uniqueness result in this case [16],[17]. Energy functionals with linear growth appear in different contexts, two classical examples being the nonparametric area integrand $f(\xi) = \sqrt{1 + \|\xi\|^2}$, which is associated with the time-dependent minimal surface equation, and the Hencky model in plasticity.

Let us summarize the contents of this book.

Chapter 1 is devoted to the study of the variational approach to image restoration based on total variation minimization subject to the constraints given by the image acquisition model. We review the model initially introduced by L. Rudin, S. Osher and E. Fatemi [175] which had, on one hand, a strong influence in the development of variational models in image denoising and restoration, and, on the other, pioneered the use of the BV model in image processing. The chapter contains the proof of the Chambolle–Lions theorem proving that the constraints can be incorporated by means of a Lagrange multiplier, thus justifying the usual numerical approach to the problem. Then we interpret the corresponding Euler–Lagrange equation in terms of partial differential equations by means of the PDE characterization of the subdifferential of the total variation. This result follows as a consequence of the results in [13] and has been presented in [48]. The approach we present here is a simple and direct approach to the characterization of the subdifferential of positively 1-homogeneous convex functionals of the gradient due to F. Alter in his unpublished work [3]. Then we display a few experiments on image restoration obtained with this model. The chapter also contains a review of the main numerical methods used in the variational approach to image restoration. We apologize in advance for any missing work.

In Chapter 2 we study the Neumann problem for the minimizing total variation flow. First we present the main existence and uniqueness results for this problem, which are essentially taken from [13]. Due to the homogeneity of the operator associated with the problem in L^p for any $p \geq 1$ we prove that the semigroup solutions are strong solutions. This, combined with the regularity results for quasi-minimizers of the perimeter, permits us to prove a regularizing effect on the level lines of the solution, a result which also holds for the solution of the restoration problem. The chapter also contains a proof that solutions of the Neumann problem stabilize as $t \to \infty$ by converging to the mean value of the initial datum.

The Cauchy problem for the total variation flow is studied in Chapter 3. The purpose of this chapter is to prove existence and uniqueness of entropy solutions for initial data in $L^1_{loc}(\mathbb{R}^N)$. This will enable us to study in later chapters the main features of the flow in $\mathbb{R}^N$, thus, dismissing the effect of boundary conditions. First, we study the flow in $L^2(\mathbb{R}^N)$. In Section 2 we prove uniqueness of entropy solutions for initial data in $L^1_{loc}(\mathbb{R}^N)$, using Kruzhkov's method of doubling variables. Then we prove existence for initial data in $L^1_{loc}(\mathbb{R}^N)$. We end up with the study of the time regularity of solutions.

Chapter 4 is devoted to a study of the asymptotic behaviour and qualitative properties of the solutions of the total variation flow in $\mathbb{R}^N$. We start by describing some numerically observed features of the flow, namely that local maxima (resp. minima) immediately decrease (resp. increase) and produce flat zones in the solution. For that we shall need some radially symmetric explicit solutions of the flow. We also note that the length of the level curves of the solutions is a decreasing function of time. Our next purpose will be to describe the extinction profile (the solution has a finite extinction time) of compactly supported solutions. This behaviour is described by a function which is the solution of an eigenvalue problem for the operator $-\mathrm{div}\left(\frac{Du}{|Du|}\right)$. The rest of the chapter is devoted to the study of explicit solutions of this eigenvalue problem in the plane. In the radial case, positive solutions can be fully characterized. Then we look for characteristic functions which are solutions of it. This permits characterization of the bounded sets of finite perimeter $\Omega \subset \mathbb{R}^2$ for which the function $u(t,x) = (1 - \frac{\mathrm{Per}(\Omega)}{|\Omega|} t)^+ \chi_\Omega(x)$ is an entropy solution of the minimizing total variation flow in $\mathbb{R}^2$. As an important by-product of the eigenvalue problem, one can obtain explicit solutions of the Rudin–Osher–Fatemi image denoising model. The results of this chapter have been taken from [13], [15], [34].

Chapter 5 is concerned with the Dirichlet problem for the total variation flow. In this case, the homogeneity of the operator is lost, and the notion of entropy solution in the sense of Kruzhkov is required to obtain a uniqueness result. Existence and time regularity of entropy solutions follow from the usual semigroup theory approach. The techniques introduced in this chapter will be the basis for results in the next two chapters dealing with more general operators. The presentation of this chapter is based on [14].

The next two chapters are devoted to a study of the Dirichlet problem for quasilinear parabolic equations whose operator is, in divergence form, the subdifferential of a Lagrangian which is convex and has linear growth in the magnitude of the gradient. More precisely, we study the Dirichlet problem in a bounded domain Ω with boundary datum $\varphi \in L^1(\partial\Omega)$, for the differential operator $-\operatorname{div} \mathbf{a}(x, Du)$, where $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$, f being a convex function of ξ with linear growth as $\|\xi\| \to \infty$. The regularity assumptions we need to impose on the Lagrangian f exclude the total variation flow, i.e., the case $f(\xi) = \|\xi\|$, which was studied in Chapter 5; but we include many examples relevant in applications, like the nonparametric area integrand and Hencky plasticity. In Chapter 6 we prove existence and uniqueness of strong solutions in $L^2(\Omega)$ using the theory of nonlinear semigroups generated by subdifferential operators. Now, to get the full strength of the abstract result derived from semigroup theory, we need to characterize the subdifferential of the energy functional associated with the problem. In Chapter 7 we prove existence and uniqueness of entropy solutions for data in $L^1(\Omega)$. Existence follows by means of Crandall–Ligget's semigroup generation theorem, while uniqueness is proved using again Kruzhkov's method of doubling variables. The results of these two chapters are essentially taken from [16] and [17], respectively.

The book finishes with three appendices in which we outline some of the main tools used in the above chapters. In the first one (Appendix A) we present without proofs the main results of nonlinear semigroup theory which is the main tool used in this text to prove existence of solutions. Due to the linear growth of the energy functionals associated with the problems studied in this monograph, the natural energy space to study them is the space of functions of bounded variation. In Appendix B we outline some of the main points of the theory of functions of bounded variation used in the previous chapters. Finally, following G. Anzelloti's paper [25], Appendix C is devoted to the main results about pairings between measures and bounded measurable functions, one of the fundamental tools of the text.

It is a pleasure to acknowledge here the debt we owe to our coauthors, namely C. Ballester, G Bellettini, J.I. Diaz and M. Novaga. This monograph could not have been written without their contribution. We would like to thank also F. Alter for permitting us to reproduce his unpublished work [3]. We are also indebted to

B. Rougé and the CNES for stimulating discussions about the restoration problem which gave us a better understanding of it, and for his kind permission to reproduce the images of Chapter 1. We thank M. Bertalmio, A. Solé and B. Rougé for providing us these experiments. Finally we are indebted with L. Rudin from Cognitech Inc. for stimulating us to work on the theoretical analysis of the total variation restoration problem which motivated the subsequent work. Thanks should also be extended to many colleagues with whom we have shared their views on image processing and PDEs, among them we would like to thank Ph. Bénilan, J. Blat, A. Chambolle, P.L. Lions, F. Malgouyres, L. Moisan, S. Moll, J.M. Morel, P. Mulet, S. Osher, G. Sapiro, S. Segura, J. Toledo and J.L. Vázquez.

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Barcelona and Valencia, December 2002

Contents

Chapter 1

Total Variation Based Image Restoration

1.1 Introduction

1.1.1 The Image Model

For the purpose of image restoration the process of image formation can be modeled in a first approximation by the formula [207]

$$u_d = Q\{\Pi(k * u) + n\}, \tag{1.1}$$

where u represents the photonic flux, k is the point spread function of the optical-captor joint apparatus, Π is a sampling operator, i.e., a Dirac comb supported by the centers of the matrix of digital sensors, n represents a random perturbation due to photonic or electronic noise, and Q is a uniform quantization operator mapping $\mathbb{R}$ to a discrete interval of values, typically $[0, 255]$.

The point spread function of the optical-captor apparatus. The optical-captor system is modeled by a convolution operator whose kernel k is called its point spread function. Indeed, both the optical system and the captor can be considered as linear and translation invariant systems, and, therefore, each of them is modeled by a convolution operator. The convolution kernel k of the joint system formed by the optics and the captor is thus the convolution of the point spread functions of both separated systems.

In CCD arrays, each detector is a flux integrator (which counts the number of photons arriving to it). Thus, its point spread function is the normalized characteristic function of a square (supposing that each detector has this geometry) $[-\frac{p}{2}, \frac{p}{2}] \times [\frac{p}{2}, \frac{p}{2}]$, i.e.,

$$k_{det}(x, y) = \frac{1}{p^2}\chi_{[-\frac{p}{2},\frac{p}{2}]\times[\frac{p}{2},\frac{p}{2}]}.$$

Its corresponding Fourier transform, also called the *modulated transfer function* of the system, is then

$$MTF_{det}(\xi_1, \xi_2) = \operatorname{sinc}(p\xi_1)\operatorname{sinc}(p\xi_2),$$

where

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}.$$

We note that we are using the Fourier transform in the form

$$F(f)(\xi) = \hat{f}(\xi) = \int_{-\infty}^{+\infty} f(y) e^{-2\pi i \xi y}\, dy. \tag{1.2}$$

The optical system has essentially two effects on the image: it projects images of the objects from the object plane to the image plane and degrades them. The degradation of the image due to the optical system makes that a light point source loses definition and appears as a blurred (small) region. This effect can be explained by the wave nature of light and its diffraction theory. We shall discard other degradation effects due to imperfections of optical systems such as lens aberrations [22]. Thus our main source of degradation will be the diffraction of the light when passing through a finite aperture: those systems are called diffraction limited systems.

A light source is called coherent if it emits light with a definite wavelength. If the emitted light is a mixture of wavelengths we say that the source is incoherent. Let us also recall that intensity of the light is given by the square of the electromagnetic field (a solution of Maxwell's equations). These two remarks will be taken into account to obtain the equations relating the electromagnetic field with the intensity field measured by the sensors.

Since we are assuming that the optical system is linear and translation invariant we know that it can be modeled by a convolution operator. Indeed, if the system is linear and translation invariant, it suffices to know the response of the system to a light point source located at the origin, which is modeled by a Dirac delta function δ, since any other light distribution could be approximated (in a weak topology) by superpositions of Dirac functions. The convolution kernel is, thus, the result of the system acting on δ.

We assume that the lens is located in an open bounded region A of a plane. The point spread function $h(x, y)$ in case of a monocromatic wave is approximately given, modulo a phase factor, by the Fourier transform of the characteristic function of the lens aperture:

$$h(x, y) = (\text{phase}) F(\chi_A(\lambda d_i \cdot, \lambda d_i \cdot),$$

where λ represents the wavelength of the light and d_i the distance from the lens to the image plane. This formula is obtained from the Maxwell equations using Kirchoff's scalar theory of diffraction and the Fraunhofer assumptions: the diffraction aperture in the screen is small compared to the distances d_i and R' from the aperture to the image plane and light source, respectively. For a detailed description of the theory of diffraction we refer to [184]. Intuitively, as the light wave arrives, each point of the aperture becomes the source of a spherical wave propagating to the image plane. After considering the above approximation, in particular that d_i is large compared to the dimensions of the aperture and the image, the point spread function h is given by ([184],[63])

$$h(x,y) = \frac{\lambda}{i} e^{\frac{ik}{2d_i}(x^2+y^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_A(\lambda d_i x', \lambda d_i y') e^{-2\pi i(xx'+yy')}\, dx'\, dy' \tag{1.3}$$

where $k = \frac{2\pi}{\lambda}$.

If we measure the light intensity, i.e., the square of the electromagnetic field, and we assume that the system is linear and translation invariant, the formula relating the light intensity emitted by the object I_0 and the light intensity measured by the optical system I is

$$I(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x-x_0, y-y_0)|^2 I_0(M^{-1}x_0, M^{-1}y_0)\, dx_0\, dy_0, \tag{1.4}$$

where M is the magnification factor, i.e., the quotient of the distance between two points of the image plane and the corresponding points in the scene, which is given by

$$M = \frac{d_i}{-z_0},$$

z_0 being the distance between the object plane and the plane of the aperture, where the origin of our coordinate system is located.

We shall write $k_{opt}(x,y) = |h(x,y)|^2$ and we call it the point spread function of the optical system. In case of a circular aperture of diameter D and incoherent light source centered around a wavelength λ, the point spread function k_{opt} is given by

$$k_{opt}(x) = \left(2\frac{J_1(\pi\frac{r}{r_0})}{\pi\frac{r}{r_0}}\right)^2 \tag{1.5}$$

where $J_1(r)$ is the Bessel function of first class and order 1, r is the radial distance computed in the image plane and

$$r_0 = \frac{\lambda d_i}{D}. \tag{1.6}$$

If the aperture is a square $[-a,a] \times [-b,b]$, then k_{opt} is given by

$$k_{opt}(x_1,x_2) = \frac{\sin^2\left(\pi\frac{x_1}{x_{01}}\right)}{\left(\pi\frac{x_1}{x_{01}}\right)^2}\frac{\sin^2\left(\pi\frac{x_2}{x_{02}}\right)}{\left(\pi\frac{x_2}{x_{02}}\right)^2} \tag{1.7}$$

where $x_{01} = \frac{\lambda d_i}{2a}$, $x_{02} = \frac{\lambda d_i}{2b}$.

The point spread function of the joint optical-captor system is the convolution of the point spread functions of both systems, i.e.,

$$k = k_{opt} * k_{det}.$$

In terms of its Fourier transforms, the modulated transforms of the optical system and detector, we have

$$MTF = MTF_{opt} MTV_{det}.$$

Noise. We shall describe the typical noise in case of a CCD array. Light is constituted by photons (quanta of light) and those photons are counted by the detector. Typically, the sensor registers light intensity by transforming the number of photons which arrive to it into an electric charge, counting the electrons which the photons take out of the atoms. This is a process of a quantum nature and therefore there are random fluctuations in the number of photons and photoelectrons on the photoactive surface of the detector. To this source of noise we have to add the thermal fluctuations of the circuits that acquire and process the signal from the detector's photoactive surface. This random thermal noise is usually described by a zero-mean white Gaussian process. The photoelectric fluctuations are more complex to describe: for low light levels, photoelectric emission is governed by Bose–Einstein statistics, which can be approximated by a Poisson distribution whose standard deviation is equal to the square root of the mean; for high light levels, the number of photoelectrons emitted (which follows a Poisson distribution) can be approximated by a Gaussian distribution which, being the limit of a Poisson process, inherits the relation between its standard deviation and its mean [22]. In a first approximation this noise is considered as spatially uncorrelated with a uniform power spectrum, thus a white noise. Finally, both sources of noise are assumed to be independent.

Taken together, both sources of noise are approximated by a Gaussian white noise, which is represented in the basic equation (1.1) by the noise term n. The average signal to noise ratio, called the SNR, can be estimated by the quotient between the signals average and the square root of the variance of the signal.

The detailed description of the noise requires a knowledge of the precise system of image acquisition. More details in the case of satellite images can be found in [172] and references therein.

The processes of image transmission and register generate other types of noise like the loss of some values or a change of the intensity value proportional to it. This could be modeled with a term η in the equation $u_d = Q\big\{\Pi(k * u) + n\big\} \cdot \eta$.

1.1.2 Image Restoration

We suppose that our image (or data) u_d is a function defined on a bounded and piecewise smooth open set D of $\mathbb{R}^N$ — typically a rectangle in $\mathbb{R}^2$. From