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J. E. JACKSON

Sphere, Spheroid and Projections for Surveyors



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Sphere, Spheroid and Projections for Surveyors

Aspects of modern land surveying

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Foreword

The early Chinese appear to have realised that the Earth was some form of spherical shape rather than flat but knowledge of this was slow to reach the Western World where the credit is usually given to Pythagoras (570–497 BC). However it was Eratosthenes in 276 BC who made the first serious attempt at quantifying the parameters. Since then enormous strides have been taken in the refinements of these parameters and the methods of measurements have become very sophisticated – including the use of orbiting satellites. It remains a fact however that there is still no way in which such a shape can be represented truly in all its aspects on a flat sheet of paper. Hence the need for map projections.

The surveyor can be very accurate in his measurements to the stars or sun or in his distances and angles on the Earth's surface but to relate them correctly to one another there is often a need to use spherical trigonometry.

Thus it is that this book aims to bring together the various facets of the surveyor's and geographer's interests that involve spherical trigonometry and the geometry of the ellipse and spheroid. The author is a very experienced practitioner in these fields and brings a new approach to the treatment of the subject as seen by a professional surveyor. It should have appeal in a number of disciplines and be of assistance to both student and practitioner.

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Portsmouth Polytechnic

Preface

The three parts of this book may be described briefly as:

I *Spherical trigonometry*: The development of formulas that may be useful to a surveyor in connection with astronomical observations, survey computations and mapping systems.

II *Spheroid geometry*: Formulas for computation of geodetic surveys on a spheroid of small eccentricity.

III *Map projections*: Methods for the precise representation of a spherical or spheroidal surface on to a plane. Problems in relating ground survey measurements to a plane coordinate system.

The main application of spherical trigonometry is in the treatment of astronomical observations which surveyors must sometimes make in order to apply correct processes of adjustment and computation to measurements made on the surface of the Earth. Spherical trigonometry has applications also in geography, cartography, navigation, geophysics and other disciplines.

Extensive survey systems for mapping purposes have always been reduced to two-dimensional geometry by representing them on a spheroidal surface, since this is the simplest form of surface that can be placed so as to be a sufficiently close fit to the somewhat irregular surface called the geoid, or mean sea level.

For the construction of maps, precise methods and formulas are required, to transform positions on the curved surface to a plane coordinate system. Such a transformation is a map projection. Many projection systems have been devised. The cartographer should have an extensive knowledge of these. National survey systems are expressed as plane coordinates of points. Land surveyors and civil engineers should know how to relate new survey

measurements to such a system, taking account of the structure of the projection in which the coordinates have been calculated.

I am grateful to Professor Gordon Petrie of the University of Glasgow for making arrangements for the fair drawings, which were made by Mrs Hadden.

Cambridge

J. E. Jackson

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Part I Spherical trigonometry

CHAPTER 1

Angles only

In any situation where we are concerned with the relative directions of a number of lines radiating from a single origin, we may think of a sphere with its centre at that origin. Each radiating line will cut the sphere at a point; if one of the lines moves, its corresponding point will trace out a curve on the sphere. Thus the geometry on the sphere is an expression of the relative positions of the radiating lines. What is significant about this is that the lines are in three-dimensional space but, so long as the linear lengths of the lines are not to be considered, the geometry of the situation is represented on the two-dimensional surface of the sphere.

We may go further and suggest that in any problem concerned only with angles and directions, it should be possible to proceed without introducing any linear quantity, that is without stating a scale.

This is what spherical trigonometry is all about. Its most direct applications in surveying are in field astronomy. To the surveyor, a star in the sky is simply an indicator of a known direction; its distance from him is enormous and irrelevant. The surveyor wants to find latitude, longitude, bearing and these are all angles.

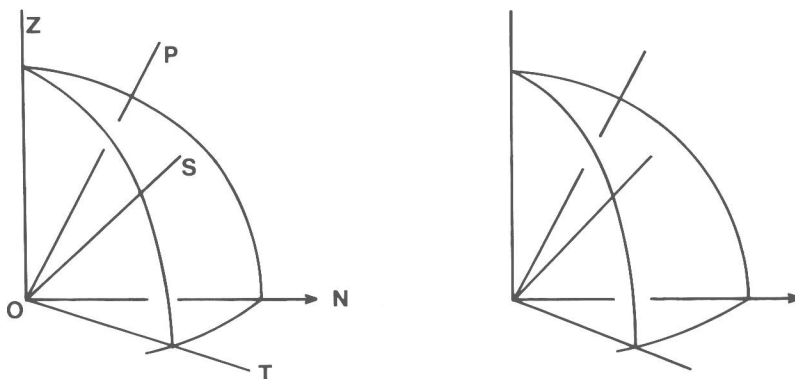
1.1 Celestial lines

In field astronomy the observer with a theodolite is concerned with three directions radiating from the centre of the instrument, point O in fig. 1.1. Direction OZ is the vertical or plumb-line, that is the direction fixed by gravity at the place where the instrument is set up. When the theodolite is correctly 'levelled' its mechanical axis of rotation coincides with this line. Another line through O is OS, the direction to a star on which the observer sets the telescope. The third direction of concern is OP, which is the line

through O and parallel to the Earth's axis of rotation. We often see photographs that have been exposed for long periods at night and they show circular tracks of stars due to the Earth's rotation. The centre of these circles plane. By definition, the pole of the sky determines the earthly direction of

Although in reality the directions OP and OS are fixed in space, the observer who is going round with the Earth senses that OZ is fixed, and it is OS that apparently moves. The rate is just perceptible in the telescope of an ordinary theodolite when the star is close to one of the lines of the diaphragm.

1.2 True north



1.1 This diagram is a stereo-pair

Referring to fig. 1.1, think of the plane through O and perpendicular to OZ. This plane is horizontal at O, and any angle measured on the graduated horizontal plate of the theodolite can be seen as an angle about O in this plane. By definition, the pole of the sky determines the earthly direction of *true north*, or *true south* as the case may be. If there were a star exactly on the Earth's axis of rotation (there is not one), the sight line of the telescope could be set on it and then the telescope could be brought down to horizontal, when it would be pointing in the direction ON, true north for an observer in the northern hemisphere.

1.3 Azimuth

Similarly, the telescope could be pointed (momentarily) to a star and then brought down to the horizontal direction of the star, OT in the figure. The angle NOT is a horizontal angle reckoned clockwise from true north. In other words it is, or rather was, the bearing or *azimuth* of the star at the moment when the telescope was set on it.

1.4 Meridian

The plane of the three lines OZ, OP and ON is the meridian plane of the observer's instrument. One side of it is east and the other side of it is west. When a star is (momentarily) in this plane it is at *transit*.

1.5 Latitude

Astronomical latitude is usually defined as the angle made by OZ with the equatorial plane: it comes to the same thing to say that the angle ZOP is the co-latitude of the observer. It follows that the angle PON is the latitude and it is the elevation of the pole of the sky above horizontal. German writers usually refer to the astronomical latitude as *pole-height*.

1.6 Altitude

Another elevation or altitude to be seen on fig. 1.1 is angle TOS, the altitude of the star. This is the only angle in the situation that can be directly measured with a theodolite. In practice, of course, the angle read on the instrument must be corrected for atmospheric refraction. The complementary angle ZOS is called the *zenith distance* and on many modern theodolites the reading of the vertical circle gives this angle directly.

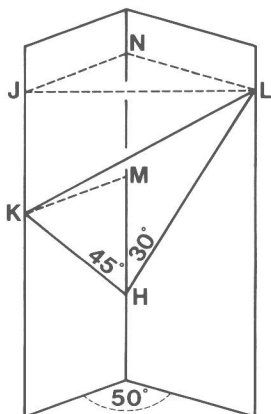
1.7 An astronomical operation

One of the practical jobs in field astronomy involves calculating the azimuth (angle NOT) of a star when a measurement of its altitude has been made. In this situation the two angles POZ (observer's co-latitude) and POS (*polar distance* of the star) are also known. Thus the relative directions of the three lines OZ, OP and OS, are known and it must be possible to calculate the azimuth. Clearly, the introduction of any linear dimension into the calculation should be unnecessary: the scale of fig. 1.1 is irrelevant.

1.8 Another kind of problem

As another example of a situation concerned only with angles, consider the simple contrived problem illustrated in fig. 1.2. A book is open so that the two opposite pages are at an angle of 50° . From a point H on the 'hinge' a line HK is drawn on one page at an angle of 45° to the hinge, and line HL is drawn on the other page at an angle of 30° to the hinge. What is the angle between HK and HL?

Evidently the answer does not depend on the physical size of the book, the lengths of HK and HL or on any other linear dimension. Nevertheless, one may be tempted to introduce a length, perhaps as a symbol or as some



1.2

arbitrary numerical value, and proceed by calculating other lengths and eventually the required angle.

Suppose that the pages are 100 mm wide.

Then the length of HK is $100 \operatorname{cosec} 45^\circ = 141.421 \text{ mm}$

and the length of HL is $100 \operatorname{cosec} 30^\circ = 200.000 \text{ mm}$

KM and LN are perpendicular to the hinge and 100 mm long.

So HM is $100 \cot 45^\circ = 100.000 \text{ mm}$

and HN is $100 \cot 30^\circ = 173.205 \text{ mm}$.

Hence MN is 73.205 mm.

Now draw NJ perpendicular to the hinge. Then NJL is an isosceles triangle with sides 100 mm and angle 50° , so the length of JL is twice $100 \sin 25^\circ$, that is 84.524 mm. JK is equal to NM. Finally, the length of KL can be calculated as the hypotenuse of the right-angled triangle LJK and the result is 111.818 mm. Now the lengths of the three sides of the plane triangle HKL are known and the angle at H can be found by a standard formula of plane trigonometry. The answer is $32^\circ 53' 55''$.