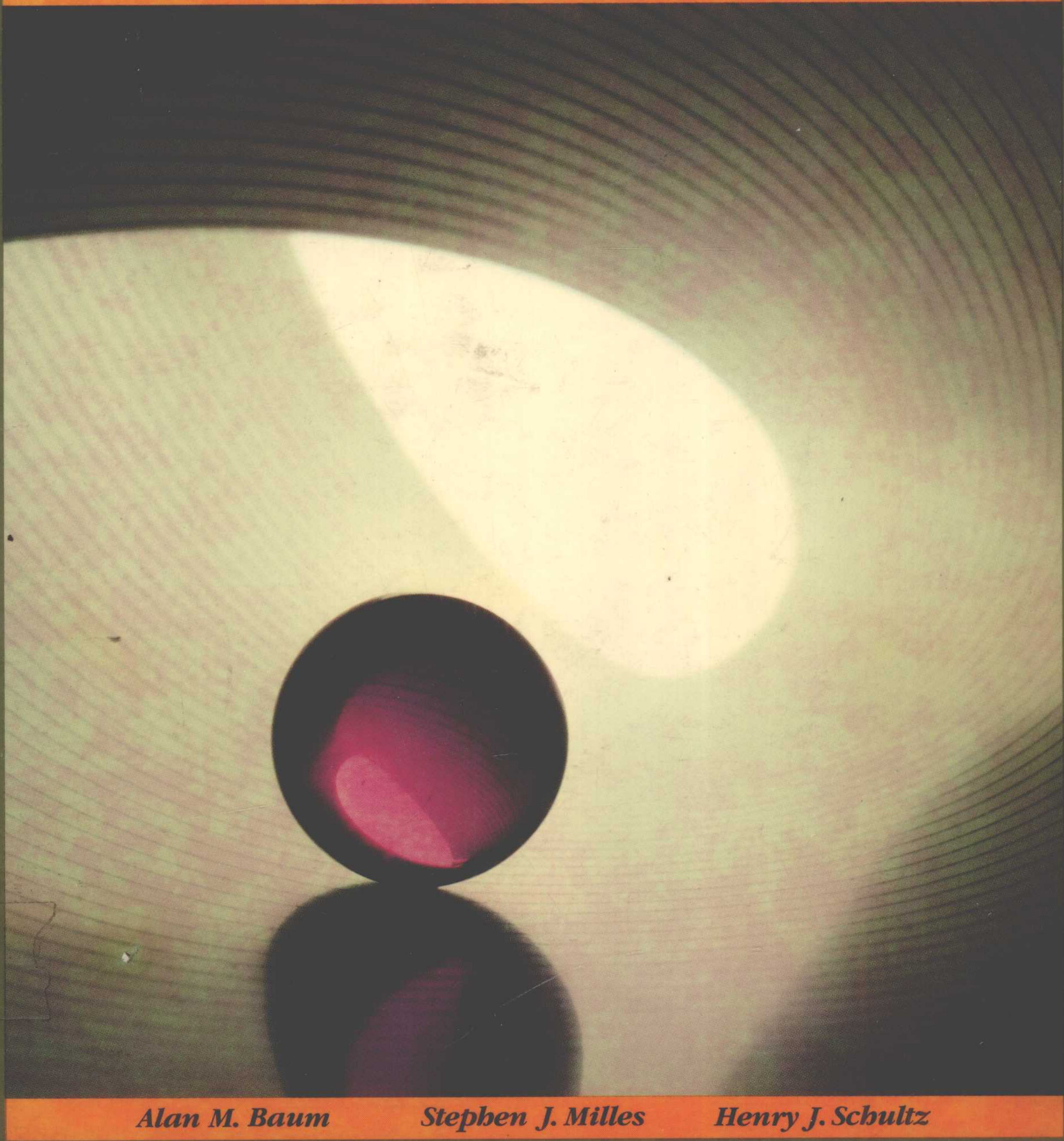
APPLIED CALCULUS



Alan M. Baum *Stephen J. Milles* *Henry J. Schultz*

APPLIED CALCULUS

Alan M. Baum
Senior Research Scientist
Computer Science Department, GM Research Laboratories

Stephen J. Milles
Chairman, Department of Mathematics and Statistics
University of Michigan/Dearborn

Henry J. Schultz
Senior Computer Applications Specialist
Lockheed Aircraft Corporation

JOHN WILEY & SONS
New York
Chichester
Brisbane
Toronto
Singapore

Cover: Designed by Edward A. Butler,
Photograph by Paul Silverman

Library of Congress Cataloging in Publication Data:

Baum, Alan.
Applied calculus.

Includes index.
1. Calculus. I. Milles, Stephen. II. Schultz, Henry.
III. Title.
QA303.B354 1985 515 84-19516

ISBN 0-471-80306-5

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

PREFACE

This book is designed to teach the techniques of calculus as they are applied to problems in business and biology, as well as the skills required to use these techniques. Throughout the text, we introduce calculus concepts through real-world examples to achieve these goals.

In our writing, we have been guided by two interrelated principles: (1) We do not serve the student well if all we teach is how to manipulate the mathematical symbols of calculus, and (2) to understand the usefulness of calculus as a problem-solving tool in business and biology, it is necessary to become fluent in the language of calculus.

Thus, we balance the motivational, descriptive, and conceptual aspects of our presentation with the manipulative drill work required to achieve a reasonable level of mastery of the material.

- New concepts are introduced by means of motivational real-world examples that demonstrate the need for the new techniques.
- Clear, detailed explanations of the material follow their introduction.
- Several worked examples of a new technique or concept are supplied, ranging from basic to more advanced.
- A large supply of exercises gives the student the opportunity to apply the new knowledge. Included are problems testing comprehension of the ideas and applications as well as the necessary, but not sufficient, technical drill exercises.
- Solutions to odd-numbered exercises appear at the back of the text. Solutions to even-numbered exercises and sample tests with solutions are contained in the accompanying instructor's manual.
- Each chapter ends with a summary reviewing the key points and a chapter test with answers.

This book is intended for a one-semester or two-quarter course; however, substantial additional material is included. At the University of Michigan—Dearborn, where the text has been used in a preliminary edition, Chapters 1 to 5 and some supplementary material on linear programming are covered in one semester. Chapters 6 and 7 are covered in the first half of a second semester, followed by supplementary material on discrete and continuous probability and statistics.

A one-semester course for students with a solid precalculus foundation might start at Chapter 2 and cover selected sections of Chapters 6 and 7. Alternatively, a two-semester course for students with only a second year of high school algebra as prerequisite, might cover the entire text with the addition of material on finite mathematics, precalculus, or other suitable topics.

In writing this text, we were fortunate to be assisted by several excellent typists, among them: Hilde McClure, Diane Patterson, Leah Long, Maria Setzler, Joyce Moss, and Sandy Flack. We also thank the many reviewers of the various versions of the manuscript for their insightful comments and suggestions. Special words of thanks are due to Mary Lequesue for her substantial work on the style and presentation of early versions of the manuscript, to Theron Shreve for his guidance through the publishing world, and to Carolyn Moore and the many professional staff members at John Wiley who worked to bring this project to a successful completion.

Alan M. Baum
Stephen J. Milles
Henry J. Schultz

PARTIAL LIST OF APPLICATIONS

CONTENTS

1

PRELIMINARIES

1.1 ALGEBRA REVIEW

We present the essential elements of algebraic manipulations here for review. Consider first the rules governing the use of exponents. Expressions of the form x^2, 10^{10}, y^{-1}, $(x^2 + 1)^{1/2}$, cm^2 all involve exponents. In the expression x^2 we call x the base and 2 the exponent. The exponent is a shorthand form for indicating the multiplication $x \cdot x$. Some multiplication results give us general rules for handling exponents. For example,

$$x^2 \cdot x^3 = (x \cdot x) \cdot (x \cdot x \cdot x) = x^5$$

$$x^3 \div x^2 = \frac{x \cdot x \cdot x}{x \cdot x} = x = x^1$$

$$(x^2)^3 = (x^2) \cdot (x^2) \cdot (x^2) = (x \cdot x) \cdot (x \cdot x) \cdot (x \cdot x) = x^6$$

$$(xy)^2 = (xy) \cdot (xy) = x^2 \cdot y^2$$

The general rules formulated for all numbers m and n are as follows.

Rules

1. $x^m \cdot x^n = x^{m+n}$
2. $x^m \div x^n = x^{m-n} \qquad x \neq 0$
3. $(x^m)^n = x^{mn}$
4. $(xy)^n = x^n y^n$
5. $x^0 = 1 \qquad x \neq 0$

To interpret (2) when $m < n$, study the following example:

$$\frac{x^2}{x^3} = \frac{x \cdot x}{x \cdot x \cdot x} = \frac{1}{x}$$

Rule (2) led us to conclude

$$\frac{x^2}{x^3} = x^{2-3} = x^{-1}$$

We use this fact as a basis for defining negative exponents. By agreement we let $x^{-1} = \frac{1}{x}$ and, in general, $x^{-n} = \frac{1}{x^n}$, for $x \neq 0$. Letting $n = -m$, we also have $x^m = \frac{1}{x^{-m}}$.

■ **EXAMPLE 1** Simplify

$$\frac{2^5 \cdot x^{-3} \cdot x^4}{2^{-1} \cdot (x^3)^2}$$

Solution:

$$\frac{2^5 \cdot x^{-3} \cdot x^4}{2^{-1} \cdot (x^3)^2} = \frac{2^5}{2^{-1}} \cdot \frac{x^{-3+4}}{x^{3 \cdot 2}} = 2^{5-(-1)} \cdot \frac{x^1}{x^6}$$

$$= 2^6 \cdot x^{1-6} = 2^6 x^{-5} = \frac{2^6}{x^5} = \frac{64}{x^5}$$ ■

In order to solve the equation $x^3 = 8$, we take cube roots of both sides, getting $x = \sqrt[3]{8} = 2$. However, rule (3) suggests that if we raise both sides of the equation $x^3 = 8$ to the power $\frac{1}{3}$, we obtain $(x^3)^{1/3} = 8^{1/3}$, or $x^{3 \cdot 1/3} = 8^{1/3}$, and finally $x = 8^{1/3}$. A reasonable interpretation for $8^{1/3}$ is then $\sqrt[3]{8} = 2$.

Definition 1 $x^{1/q} = \sqrt[q]{x}$, for q a positive integer, $x \geq 0$.

This idea allows us to extend the exponent concept to other fractional exponents.

Definition 2 $x^{p/q} = (x^{1/q})^p = (x^p)^{1/q}$, for p and q integers, q positive.

■ EXAMPLE 2

$$64^{2/3} = (64^{1/3})^2 = 4^2 = 16$$ ■

■ EXAMPLE 3

$$(x^4y)^{-5/2} = ((x^4y)^{1/2})^{-5} = (x^2y^{1/2})^{-5} = \frac{1}{(x^2y^{1/2})^5} = \frac{1}{x^{10}y^{5/2}}$$ ■

By common agreement $\sqrt{4} = +2$ only, even though $(-2)^2 = 4$.

The negative square root of 4 is written $-\sqrt{4}$; if we desire both we will write $\pm\sqrt{4}$.

We apply the same rules (1) to (4) to expressions involving irrational numbers as exponents, for example, x^π, $2^{\sqrt{2}}$.

■ EXAMPLE 4

$$\frac{a^{\sqrt{2}}b}{ab^4} \cdot (bc)^{2\pi} = a^{(\sqrt{2}-1)}b^{-3} \cdot b^{2\pi}c^{2\pi} = a^{(\sqrt{2}-1)}b^{(2\pi-3)}c^{2\pi}$$ ■

■ EXAMPLE 5 Solve the equation

$$(x^3 + 1)^{-1/2} = \tfrac{1}{3}$$

Solution: Raise both sides to the power -2 and get

$$[(x^3 + 1)^{-1/2}]^{-2} = (\tfrac{1}{3})^{-2}$$

$$x^3 + 1 = \frac{1}{(\frac{1}{3})^2} = 9$$

Then $x^3 = 8$, so $x = 8^{1/3} = 2$. ■

Notice that *no rule exists* that allows $(x^3 + 1)^{-1/2} = (x^3)^{-1/2} + 1^{-1/2}$ *This is not a legitimate algebraic operation. Only* when $n = +1$ is $(x + y)^n = x^n + y^n$ true. We can test this observation further: $\sqrt{4 + 5} = (4 + 5)^{1/2} = 9^{1/2} = 3$; however, $4^{1/2} + 5^{1/2} = 2 + \sqrt{5} \approx 4.436 \neq 3$. And $(2 + 4)^{-1} = 1/(2 + 4) = \frac{1}{6}$; however, $2^{-1} + 4^{-1} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \neq \frac{1}{6}$.

When we multiply expressions of the form $(x + y)^2$, or more generally a product

$(a + b)(c + d)$, we use the fact that multiplication distributes over addition to compute the product:

$$(a + b) \cdot (c + d) = a \cdot (c + d) + b \cdot (c + d) = ac + ad + bc + bd$$

In particular

$$(x + y)^2 = x^2 + xy + yx + y^2 = x^2 + 2xy + y^2$$

■ **EXAMPLE 6** Multiply and simplify the following:

(a) $(x^2 + y)(x^2 - y)$
(b) $(2a + 3)(a^2 - a^{-1} + 1)$

Solution:

(a) $(x^2 + y)(x^2 - y) = x^2 \cdot x^2 + x^2(-y) + y \cdot x^2 - y \cdot y = x^4 - y^2$
(b) $(2a + 3)(a^2 - a^{-1} + 1) = 2a \cdot a^2 - 2a \cdot a^{-1} + 2a \cdot 1 + 3 \cdot a^2 - 3 \cdot a^{-1} + 3 \cdot 1$

$$= 2a^3 - 2 + 2a + 3a^2 - \frac{3}{a} + 3$$

$$= 2a^3 + 3a^2 + 2a + 1 - \frac{3}{a}$$ ■

It is also useful to factor expressions.

■ **EXAMPLE 7** Factor $x^2 - 2x - 3$.

Solution: We want to find expressions of the form $(x + a)$ and $(x + b)$ which, when multiplied together, give $x^2 - 2x - 3$. Since $(x + a)(x + b) = x^2 + (a + b)x + ab$, the x^2 terms match. We now try to select a and b, so that $a + b = -2$ and $ab = -3$. We see that $a = 1$, $b = -3$ works and the correct factorization is

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

Multiplication checks our work. ■

■ **EXAMPLE 8** Factor $x^3 + 7x^2 + 10x$.

Solution: First we note that x is a factor common to all three terms. Thus $x^3 + 7x^2 + 10x = x(x^2 + 7x + 10)$. We now factor the quadratic so that $x^2 + 7x + 10 = (x + 2) \cdot (x + 5)$ and therefore

$$x^3 + 7x^2 + 10x = x(x + 2)(x + 5)$$ ■

A very common factorization worth remembering is the difference of two squares

$$x^2 - y^2 = (x + y)(x - y)$$

■ **EXAMPLE 9**

(a) Factor $x^2 - 25$.
(b) Factor $(x + y)^2 - z^2$.

Solution:

(a) $(x^2 - 25) = (x + 5)(x - 5)$
(b) $(x + y)^2 - z^2 = [(x + y) + z][(x + y) - z]$ ■

Another method of factoring makes use of the *quadratic formula*, which states: $ax^2 + bx + c = 0$ has solutions (roots)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

whenever $b^2 - 4ac \geq 0$. (If $b^2 - 4ac < 0$, there are no real roots.)

Returning to Example 7, we see that the roots of

$$x^2 - 2x - 3 = 0 \qquad \text{are} \qquad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-3)}}{2(1)} = 3, -1$$

Having the roots, we may construct the factors $(x - 3)$ and $[x - (-1)] = (x + 1)$. This is an application of a general result called the *Fundamental Theorem of Algebra*, which states:

The expression $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ has the factorization $c_n(x - r_1)(x - r_2) \ldots (x - r_n)$, where the numbers $r_1, r_2, \ldots, r_n$ are the n roots of the equation $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$.

■ **EXAMPLE 10** Factor $2x^2 - 3x + 1$.

Solution: The roots of $2x^2 - 3x + 1 = 0$ are

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 2}}{2 \cdot 2} = \frac{3 \pm 1}{4} = 1, \frac{1}{2}$$

by the quadratic formula. By the Fundamental Theorem of Algebra the factorization is

$$2x^2 - 3x + 1 = 2(x - \tfrac{1}{2})(x - 1) = (2x - 1)(x - 1)$$ ■

■ **EXAMPLE 11** Factor $x^3 - 2x^2 - 5x + 6$.

Solution: We must guess a root of $x^3 - 2x^2 - 5x + 6 = 0$. Clearly $x = 0$ is not a root. We try $x = 1$ and find it is a root, since $1 - 2 - 5 + 6 = 0$. Therefore one of the factors is $(x - 1)$. Instead of trying to guess the other roots, we divide $x^3 - 2x^2 - 5x + 6$ by $(x - 1)$.

$$\begin{array}{r}
x^2 - x - 6 \\
x - 1\overline{)x^3 - 2x^2 - 5x + 6} \\
\underline{x^3 - x^2 \qquad\qquad} \\
-x^2 - 5x \qquad \\
\underline{-x^2 + x \qquad} \\
-6x + 6 \\
\underline{-6x + 6} \\
0
\end{array}$$

Therefore $x^3 - 2x^2 - 5x + 6 = (x - 1)(x^2 - x - 6)$, and factoring the quadratic gives $(x - 1)(x - 3)(x + 2)$ as the factorization. ■

Often we can mathematically express a relationship between variables. For example, if the total cost, C, of golf balls is directly proportional to the number purchased, x, we can express this as $C = kx$ for some number k.

Definition 3 The variable y is *directly proportional* to the variable x if there is some number k such that $y = kx$.

To find k we use some specific information relating cost and number, such as one dozen golf balls cost $8.

■ **EXAMPLE 12** The amount of money received as dividends from a company is directly proportional to the number of shares of stock owned. If Mary owns 100 shares and receives a dividend of $35, how much will Tom receive if he owns 600 shares? How many shares does Linda own if she receives a dividend of $7?

Solution: Let d denote the amount of the dividend and let s denote the number of shares. The fact that the amount of the dividend is directly proportional to the number of shares means that there is a constant k such that $d = ks$. Since, in Mary's case, $d =$ \$35 when $s = 100$ shares, we find

$$\$35 = k \cdot (100 \text{ shares})$$

or

$$k = \$0.35/\text{share}$$

Thus Tom receives

$$\begin{aligned} d &= (\$0.35/\text{share}) \cdot (600 \text{ shares}) \\ &= \$210 \end{aligned}$$

To find the number of shares Linda owns we set

$$\$7 = (\$0.35/\text{share}) \cdot s$$

and solving, get $s = 20$ shares. ■

■ **EXAMPLE 13** The rate at which chemical A is converted into chemical B in a reaction is directly proportional to the amount of chemical A present. Let r (g/min) be the rate at which A is converted to B and let a (g) be the amount of A present. The statement that r is directly proportional to a allows us to write $r = ka$ for some constant k. If A is being converted to B at the rate of 2 g/min when there are 40 g of A present, at what rate is A being converted to B when there are 20 g of A present?

Solution: Since $r = 2$ g/min when $a = 40$ g, we find 2 g/min $= k \cdot 40$ g so that $k = \frac{1}{20}$/min. Then the rate of conversion when there are 20 g of A present is given by $r = (\frac{1}{20}/\text{min})(20 \text{ g}) = 1$ g/min. ■

Note that halving the number of grams of A present (from 40 to 20) results in halving the conversion rate r (from 2 to 1). Analogously, tripling the amount of A present would result in tripling the conversion rate. This relationship characterizes direct proportionality.

In marketing, sales volume is often increased if we decrease selling price. If x denotes the number of items to be sold and p the price per item, then $p = D(x)$ is called a *demand function*, giving the relationship between the number sold and the price. In general, we expect p to decrease as x increases. Suppose $p = D(x) = 20{,}000/x$. Then, in order to sell 10,000 items, we set the price at \$2 each (20,000/10,000); in order to sell 50,000 items we must reduce the price to \$0.40 each.

A five-fold increase in sales results when we reduce the price by a factor of 5; that is, since $50{,}000 = 5 \times 10{,}000$, the price must be $\$0.40 = \frac{1}{5} \times \$2$.

We say that variables x and p are inversely proportional. In general we have the following definition.

Definition 4 The variable y is *inversely proportional* to the variable x if there is some number k such that $y = k/x$.

In our problem $k = 20{,}000$.

Saying that y is inversely proportional to x is the same as saying that y is directly proportional to $1/x$.

When two variables are inversely proportional, their product is a constant, that is, $xy = k$. Therefore, as x gets bigger y must get smaller.

■ **EXAMPLE 14** The equation $PV = KnT$ relates the pressure P, volume V, and temperature T of a gas to the number of molecules n of the gas present where K is a constant. Solving for P, we find $P = (KnT)/V$, from which we conclude that P is directly proportional to n and T and inversely proportional to V. With n and V fixed, doubling the temperature of the gas doubles its pressure. With n and T fixed, doubling the volume halves the pressure. With n fixed, what happens to the pressure if we double both the temperature and the volume? It remains the same. ■

Exercise Set 1.1

1.1.1. Using rules of exponents (1) to (4), prove rule (5). Are there any restrictions on the values of x for which (5) holds true?

In Exercises 1.1.2 to 1.1.9, simplify the expression.

1.1.2. $\dfrac{x^7\, y}{x^{-3}y^2}$.

1.1.3. $\dfrac{(xy)^2 \cdot (x^3y^2)^2}{x^4y^6}$.

1.1.4. $\sqrt[3]{x} \cdot \sqrt[4]{x}$.

1.1.5. $\sqrt{x+1}\,(x+1)^{-5/2}$.

1.1.6. $\dfrac{(3x)^2 \cdot (2y)^{-3}}{x^3 \cdot y^{3/2}}$.

1.1.7. $(\sqrt[5]{x})^{10} \div x^{-2}$.

1.1.8. $\dfrac{(4x)^{1/2}\,(9y)^{-2}}{12x^{-7/2}}$.

1.1.9. $\dfrac{96(x^2y^{1/2})^{-1/4}}{24\,(\sqrt{xy})^8}$.

In Exercises 1.1.10 to 1.1.15, write as a single fraction.

1.1.10. $\frac{1}{a} + \frac{1}{b}$.

1.1.11. $\frac{1}{x} + y^{-2}$.

1.1.12. $a^{-1/2} + b^{-1/2}$.

1.1.13. $a^{-3} + 1$.

1.1.14. $\frac{x^{-1}}{y} + \frac{y^{-1}}{x}$.

1.1.15. $\frac{a^{-1}}{a + b} + \frac{b^{-1}}{a + b}$.

In Exercises 1.1.16 to 1.1.23, multiply and collect like terms.

1.1.16. $(a + b + b^2)(a - b^2)$.

1.1.17. $\left(\frac{1}{x} + \frac{1}{y}\right)(x - y)$.

1.1.18. $(2 + 3x + \sqrt{x})(x - \sqrt{x + 1})$.

1.1.19. $(3x + 2y)(x - 5y)$.

1.1.20. $(\sqrt{x - 2} + \sqrt{x + 1})(\sqrt{x - 2} - \sqrt{x + 1})$.

1.1.21. $(x^2 - x + 1)(x^2 + x + 2)$.

1.1.22. $(3x^2 + 5x)(x^2 - 2x + 1)$.

1.1.23. $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y} + \sqrt{x + y})$.

In Exercises 1.1.24 to 1.1.34, factor.

1.1.24. $x^2 - 4x - 5$.

1.1.25. $x^2 + 4x + 3$.

1.1.26. $2x^2 + 5x - 12$.

1.1.27. $r^4 - 3r^2 - 4$.
(*Hint:* let $x = r^2$).

1.1.28. $t^2 - 2t\sqrt{y} + y$.

1.1.29. $x^2 + 8x + 12$.

1.1.30. $x^4 - y^4$.

1.1.31. $x^3 + 4x^2 - x - 4$.

1.1.32. $x^3 - 6x^2 + 8x$.

1.1.33. $x^3 + 6x^2 + 9x + 4$.

1.1.34. $x^4 - 2x^3 + 2x - 1$.

In Exercises 1.1.35 to 1.1.41, solve using the quadratic formula.

1.1.35. $x^2 + 3x - 10 = 0$.

1.1.36. $x^2 - \frac{x}{2} + \frac{1}{16} = 0$.

1.1.37. $x^2 + x - 1 = 0$.

1.1.38. $3x^2 - 2x - 1 = 0$.

1.1.39. $x^2 + 3x + 1 = 0$.

1.1.40. $2x^2 - 8x + 3 = 0$.

1.1.41. $\frac{-x^2}{2} - 3x + 2 = 0$.

1.1.42. A bank compounds interest semiannually and wishes to select an interest rate that will be equivalent to 6 percent simple interest (applied at the end of the year). If it selects interest rate r, then after one year an initial deposit A will have grown to $A[1 + (r/2)]^2$. At 6 percent simple interest, A would have grown to $A(1 + 0.06)$ after one year. Find r. (We say that the interest rate r compounded semiannually has an *effective annual yield* of 6 percent.)

1.1.43. Find the total revenue for the example leading to Definition 4, where revenue is the price per item times the number of items sold. In what ways is the demand function in that example unrealistic?