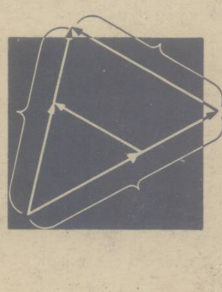
RICHARD E. JOHNSON

Vector Algebra



VOLUME FOUR

THE PRINDLE, WEBER & SCHMIDT
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Vector Algebra

RICHARD E. JOHNSON

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VECTOR ALGEBRA

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THE PRINDLE, WEBER & SCHMIDT
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Preface

Finite-dimensional vector spaces are particularly suited as the first topic for study in abstract Algebra. One reason for this is that the theory is straightforward without being trivial, and it leads to a complete description of such algebraic systems. Another reason is that vector spaces are encountered in all branches of mathematics, from analysis through geometry and topology. Finally, there are concrete examples of vector spaces available to illustrate the theory as well as to prepare the student for applications in physics, engineering, and other sciences.

Throughout the book, the underlying field of scalars is assumed to be the real number field **R**. However, in almost all sections of the book any other field F could be substituted for **R** without necessitating any changes.

The simple properties of vector spaces and their subspaces are given in the first chapter. The two principal examples of vector spaces, geometric vector spaces and the space of n-tuples of real numbers, are also presented in this chapter so that they might be used throughout the book.

Most of the theory is in Chapter 2, where the invariance of the dimension of a vector space is proved. Inner products are discussed in Chapter 3, and the existence of a normal orthogonal basis is proved for finite-dimensional vector spaces.

The very useful cross product operation in geometric vector spaces is investigated in Chapter 4. Then applications are made to the finding of equations of lines and planes in space.

In an appendix, fields in general and ordered fields in particular are defined. There is also a short section on coordinate systems for lines, planes, and space. Vector spaces over the complex number field are touched on briefly in another section. The final section relates the algebra of vectors to the quaternion algebra of Hamilton.

It is hoped that the book might prove useful in a variety of ways. In the first place, the material is detailed and amply illustrated for individual study. Thus, the student of physics or mathematics could use the book for supplementary reading on vectors. Another possible use is as a text in a senior high school course on abstract algebra. Finally, this book, along with Linear Algebra in the Prindle, Weber & Schmidt Complementary Series could serve as texts in a college course on vectors and linear algebras.

RICHARD E. JOHNSON

Contents

Chapter One

Vector Spaces

1. DEFINITION

A vector is an element of a vector space. In turn, a vector space is a set of objects, called vectors, which is closed under operations of addition and scalar multiplication and which satisfies certain algebraic laws. A precise definition of a vector space is given below.

It is worthwhile to study vector spaces for the reason that many of the algebraic systems encountered in applications of mathematics are in essence vector spaces. By studying general vector spaces, without regard to the nature of the elements, we can develop the properties common to all vector spaces.

Throughout this book the set of all real numbers will be denoted by $\mathbf{R}$. There are two basic operations in $\mathbf{R}$, addition and multiplication, and a basic order relation of greater than or equal to. Of course, $\mathbf{R}$ also has operations of subtraction and division, and order relations of greater than, less than or equal to, and less than. With respect to its operations and relations, $\mathbf{R}$ is an ordered field as defined in the Appendix.

1.1 DEFINITION OF A VECTOR SPACE. A vector space consists of a set V, an operation of addition in V, and an operation of scalar multiplication of V by $\mathbf{R}$. Addition in V has the following properties:

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$. (*Commutative law*)
(2) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. (*Associative law*)
(3) There exists an element $\mathbf{0}$ in V such that
$\mathbf{0} + \mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$. (*Identity element*)
(4) Associated with each $\mathbf{x}$ in V is an element $-\mathbf{x}$ in V such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in V$. (*Inverse elements*)

For each $a \in \mathsf{R}$ and $\mathbf{x} \in V$, the scalar product of $\mathbf{x}$ by a is a unique element of V denoted by $a\mathbf{x}$. Scalar multiplication has the following properties:

(5) $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ for all $a \in \mathsf{R}$, $\mathbf{x}, \mathbf{y} \in V$. (*Distributive law*)
(6) $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$ for all $a, b \in \mathsf{R}$, $\mathbf{x} \in V$. (*Distributive law*)
(7) $(ab)\mathbf{x} = a(b\mathbf{x})$ for all $a, b \in \mathsf{R}$, $\mathbf{x} \in V$. (*Associative law*)
(8) $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$. (*Identity element*)

The eight properties of addition and scalar multiplication listed above are quite familiar to all of us. Thus properties (1)–(4) are enjoyed by addition in R and properties (5)–(8) are enjoyed by multiplication in R, if we consider set V as being R.

We shall call elements of V *vectors* and those of R *scalars*. Vectors are denoted by boldface letters to clearly distinguish them from scalars. Usually, we denote scalars by the first few letters of the alphabet and vectors by the last few. In writing symbols for vectors, you might wish to put an arrow over the symbol to indicate that it denotes a vector.

Additional properties of a vector space may be derived from the eight defining ones. For example, we might expect the following properties to hold since corresponding ones hold in a field. Let V be a vector space, $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, and $a \in \mathsf{R}$.

1.2 If $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$ or $\mathbf{z} + \mathbf{x} = \mathbf{z} + \mathbf{y}$, then $\mathbf{x} = \mathbf{y}$. (*Cancellation law*)

1.3 $$a\mathbf{x} = \mathbf{0} \text{ if and only if either } a = 0 \text{ or } \mathbf{x} = \mathbf{0}.$$

1.4 $$-(a\mathbf{x}) = (-a)\mathbf{x} = a(-\mathbf{x}).$$

Proof of 1.2: If $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$, then

$$(\mathbf{x} + \mathbf{z}) + (-\mathbf{z}) = (\mathbf{y} + \mathbf{z}) + (-\mathbf{z})$$

$$\mathbf{x} + [\mathbf{z} + (-\mathbf{z})] = \mathbf{y} + [\mathbf{z} + (-\mathbf{z})] \qquad \text{(Assoc. law)}$$

$$\mathbf{x} + \mathbf{0} = \mathbf{y} + \mathbf{0} \qquad \text{(Inverse el.)}$$

$$\mathbf{x} = \mathbf{y} \qquad \text{(Identity el.).}$$

If $\mathbf{z} + \mathbf{x} = \mathbf{z} + \mathbf{y}$, then $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z}$ by the commutative law and $\mathbf{x} = \mathbf{y}$ by the proof above.

Proof of 1.3: This property states that $a\mathbf{x} = \mathbf{0}$ if and only if a is the zero scalar or $\mathbf{x}$ is the zero vector. If $a = 0$, the zero element of R, then $0 = 0 + 0$ and $0\mathbf{x} = (0 + 0)\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$ by 1.1(6). Since $\mathbf{0} + 0\mathbf{x} = 0\mathbf{x}$ by 1.1(3), we have

$$\mathbf{0} + 0\mathbf{x} = 0\mathbf{x} + 0\mathbf{x}$$

and

$$\mathbf{0} = 0\mathbf{x}$$

by 1.2. A similar argument shows that $a\mathbf{0} = \mathbf{0}$.

Conversely, assume that $a \in$ R and $\mathbf{x} \in V$ are such that $a\mathbf{x} = \mathbf{0}$. If $a \neq 0$, then a^{-1} exists and

$$\mathbf{x} = 1\mathbf{x} = (a^{-1}a)\mathbf{x} = a^{-1}(a\mathbf{x}) = a^{-1}\mathbf{0} = \mathbf{0}.$$

If $a = 0$, then $a\mathbf{x} = \mathbf{0}$ by the proof above. This proves 1.3.

Proof of 1.4: By 1.1(4), $a\mathbf{x} + [-(a\mathbf{x})] = \mathbf{0}$, whereas by 1.1(7) and 1.3, $\mathbf{0} = 0\mathbf{x} = [a + (-a)]\mathbf{x} = a\mathbf{x} + (-a)\mathbf{x}$. Therefore

$$a\mathbf{x} + [-(a\mathbf{x})] = a\mathbf{x} + (-a)\mathbf{x}$$

and

$$-(a\mathbf{x}) = (-a)\mathbf{x}$$

by the cancellation law. A similar proof shows that $-(a\mathbf{x}) = a(-\mathbf{x})$. This proves 1.4.

The operation of subtraction in V is defined as follows:

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in V.$$

It is easily shown that vector subtraction has properties similar to those of subtraction in R. For example,

$$a(\mathbf{x} - \mathbf{y}) = a\mathbf{x} - a\mathbf{y} \text{ for all } \mathbf{x}, \mathbf{y} \in V, a \in \mathrm{R},$$

$$-(\mathbf{x} - \mathbf{y}) = (-\mathbf{x}) + \mathbf{y} \text{ for all } \mathbf{x}, \mathbf{y} \in V.$$

Henceforth we shall assume that the reader is familiar with these properties.

EXERCISES

In the following exercises V is assumed to be a vector space.

1. Prove that $a\mathbf{0} = \mathbf{0}$ for all $a \in$ R (part of 1.3).

2. Prove that $-(a\mathbf{x}) = a(-\mathbf{x})$ for all $a \in \mathsf{R}$ and $\mathbf{x} \in V$ (part of 1.4).
3. State and prove a cancellation law for scalar multiplication.
4. Prove that $a(\mathbf{x} - \mathbf{y}) = a\mathbf{x} - a\mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in V$, $a \in \mathsf{R}$.
5. Prove that $-(\mathbf{x} - \mathbf{y}) = (-\mathbf{x}) + \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in V$.

2. SUBSPACES

If V is a vector space and S is a nonempty subset of V which is closed under addition and scalar multiplication (i.e., $\mathbf{x} + \mathbf{y}$ and $a\mathbf{x}$ are in S for all $\mathbf{x}, \mathbf{y} \in S$, $a \in \mathsf{R}$), then S is a vector space in its own right. Thus $\mathbf{0} \in S$ since $\mathbf{0} = 0\mathbf{x}$ for any $\mathbf{x} \in S$. If $\mathbf{x} \in S$, then $-\mathbf{x} \in S$ also, since $-\mathbf{x} = -(1\mathbf{x}) = (-1)\mathbf{x}$. It is now clear that 1.1(1)–(8) hold for S as well as V. We call S a *subspace* of V.

For each $\mathbf{x} \in V$ the set of all scalar multiples of $\mathbf{x}$ is denoted by $\mathsf{R}\mathbf{x}$,

$$\mathsf{R}\mathbf{x} = \{a\mathbf{x} \mid a \in \mathsf{R}\}.$$

Since

$$a\mathbf{x} + b\mathbf{x} = (a + b)\mathbf{x} \quad \text{and} \quad b(a\mathbf{x}) = (ba)\mathbf{x},$$

the set $\mathsf{R}\mathbf{x}$ is closed under addition and scalar multiplication and is therefore a subspace of V. The set of scalar multiples of $\mathbf{0}$ is simply $\{\mathbf{0}\}$. Thus $\{\mathbf{0}\}$ is a subspace of V; in fact, $\{\mathbf{0}\}$ is the least subspace of V in the sense that it is contained in every other subspace of V. Trivially, V is a subspace of itself according to our definition.

We call a subspace S of V *proper* if $S \neq \{\mathbf{0}\}$ and $S \neq V$. If $\mathbf{x} \in V$, $\mathbf{x} \neq \mathbf{0}$, then $\mathsf{R}\mathbf{x}$ is a minimal nonzero subspace. For if S is a proper subspace of V and $S \subset \mathsf{R}\mathbf{x}$, and if $\mathbf{y} \in S$, $\mathbf{y} \neq \mathbf{0}$, then $\mathbf{y} = a\mathbf{x}$ for some nonzero $a \in \mathsf{R}$. Hence $(ba^{-1})\mathbf{y} = (ba^{-1})a\mathbf{x} = b\mathbf{x}$ is in S for every $b \in \mathsf{R}$ and $\mathsf{R}\mathbf{x} \subset S$. It follows that $S = \mathsf{R}\mathbf{x}$. The subspace $\mathsf{R}\mathbf{x}$ is proper unless $V = \mathsf{R}\mathbf{x}$, a rather uninteresting possibility.

If S_1 and S_2 are subspaces of a vector space V, then so is their *intersection*,

$$S_1 \cap S_2 = \{\mathbf{x} \in V \mid \mathbf{x} \in S_1 \text{ and } \mathbf{x} \in S_2\}.$$

For if $\mathbf{x}$ and $\mathbf{y}$ are in both S_1 and S_2, then so are $\mathbf{x} + \mathbf{y}$ and $a\mathbf{x}$ for every $a \in \mathsf{R}$.

More generally, if $\{S_1, S_2, \cdots, S_n\}$ is any finite set of subspaces of V, then their intersection $S_1 \cap S_2 \cap \cdots \cap S_n$ also is a subspace of V. The notation

$$\bigcap_{i=1}^{n} S_i$$

is often used for this intersection. Evidently $S_1 \cap S_2 \cap \cdots \cap S_n$ is the *largest* subspace of V contained in all the subspaces $S_1, S_2, \cdots, S_n$.

Another useful way of forming a subspace from two given subspaces S_1 and S_2 of a vector space V is to take their *sum:*

$$S_1 + S_2 = \{\mathbf{x}_1 + \mathbf{x}_2 \mid \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2\}.$$

To show that $S_1 + S_2$ actually is a subspace of V, let $\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}_1 + \mathbf{y}_2 \in S_1 + S_2$, where $\mathbf{x}_i, \mathbf{y}_i \in S_i$, and $a \in \mathsf{R}$. Then

$$(\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{y}_1) + (\mathbf{x}_2 + \mathbf{y}_2) \in S_1 + S_2,$$

$$a(\mathbf{x}_1 + \mathbf{x}_2) = a\mathbf{x}_1 + a\mathbf{x}_2 \in S_1 + S_2.$$

Thus $S_1 + S_2$ is a subspace of V because it is closed under addition and scalar multiplication.

We can similarly form the sum of n subspaces $S_1, S_2, \cdots, S_n$ of V:

$$S_1 + S_2 + \cdots + S_n = \{\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n \mid \mathbf{x}_i \in S_i\}.$$

The sigma notation

$$\sum_{i=1}^{n} S_i$$

is often used for this sum. Since $\mathbf{0} \in S_i$ for each i, $\mathbf{x}_1 + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{x}_1 \in S_1 + S_2 + \cdots + S_n$ for each $\mathbf{x}_1 \in S_1$. That is, $S_1 \subset S_1 + S_2 + \cdots + S_n$. Similarly,

$$S_j \subset \sum_{i=1}^{n} S_i \text{ for } j = 1, 2, \cdots, n.$$

It should be clear that $S_1 + S_2 + \cdots + S_n$ is the *least* subspace of V containing all the subspaces $S_1, S_2, \cdots, S_n$.

If $\mathbf{x}, \mathbf{y} \in V$, then either $\mathsf{R}\mathbf{x} \cap \mathsf{R}\mathbf{y} = \{\mathbf{0}\}$ or $\mathsf{R}\mathbf{x} = \mathsf{R}\mathbf{y}$. On the other hand,

$$\mathsf{R}\mathbf{x} + \mathsf{R}\mathbf{y} = \{a\mathbf{x} + b\mathbf{y} \mid a, b \in \mathsf{R}\}.$$

More generally, for any $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n \in V$,

$$\sum_{i=1}^{n} \mathsf{R}\mathbf{x}_i = \left\{ \sum_{i=1}^{n} a_i \mathbf{x}_i \mid a_i \in \mathsf{R} \right\}.$$

We call $S = \sum_{i=1}^{n} \mathsf{R}\mathbf{x}_i$ the subspace of V *spanned* by the vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$. Each vector of S is said to be a *linear combination* of the n vectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$.

EXERCISES

1. If S_1, S_2, and S_3 are proper subspaces of V such that $S_1 \cap (S_2 + S_3) = \{\mathbf{0}\}$ and $S_2 \cap S_3 = \{\mathbf{0}\}$, then prove that $S_2 \cap (S_1 + S_3) = \{\mathbf{0}\}$.
2. Let $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ be nonzero vectors which generate a vector space V. If $\mathbf{y} \in V$, $\mathbf{y} \neq \mathbf{0}$, prove that $\mathbf{y}$ together with some two of the vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ generate V.
3. Generalize Exercise 2 from 3 to n vectors.
4. If S_1 and S_2 are subspaces of V such that $S_1 \supset S_2$, then prove that $S_1 \cap (S_2 + S) = S_2 + (S_1 \cap S)$ for every subspace S of V. (This is called the *modular law.*)

3. GEOMETRIC VECTORS

For many centuries physicists have used directed line segments to represent forces, velocities, accelerations, and other entities having both magnitude and direction. We shall describe in this section the space of geometric vectors, or directed line segments, used by physicists.

Consider a Euclidean plane, denoted by E_2, made up of points and lines satisfying the postulates of Euclidean geometry. Every ordered pair (A, B) of points in E_2 determines a *segment* with endpoints A and B and a *direction* from the *initial point* A to the *terminal point* B. We call such a directed segment a *vector* in E_2, and denote it by

$$\mathbf{AB}.$$

The set of all vectors in E_2 is denoted by

$$V_2.$$

It will be convenient to have a coordinate system in E_2 (see the Appendix) so that each vector **AB** has a length denoted by

$$|\mathbf{AB}|$$

and defined to be $d(A, B)$, the distance between points A and B.

We shall consider the vectors in V_2 to be *free vectors;* i.e., two vectors **AB** and **CD** will be considered to be equal,

$$\mathbf{AB} = \mathbf{CD},$$

if and only if either $|\mathbf{AB}| = |\mathbf{CD}| = 0$ (i.e., $A = B$ and $C = D$) or $|\mathbf{AB}| = |\mathbf{CD}| \neq 0$ and **AB**, **CD** are parallel and directed in the same way (Fig. 1.1). Thus the position of each vector in the plane is immaterial; only its length and direction are important. If we wish, all vectors can be assumed to have the same initial point.

The sum of two vectors **AB** and **BC** is defined to be **AC**,

$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC},$$

as shown in Fig. 1.2. If the two vectors **AB** and **AD** have the same initial point, then their sum is **AC**,

$$\mathbf{AB} + \mathbf{AD} = \mathbf{AC},$$

where **AC** is the diagonal of the parallelogram having **AB** and **AD** as two of its sides (Fig. 1.3). This is true because $\mathbf{AD} = \mathbf{BC}$.

If the points A, B, and C are collinear, then there is no parallelogram associated with the sum $\mathbf{AB} + \mathbf{BC}$. Two possible cases are shown

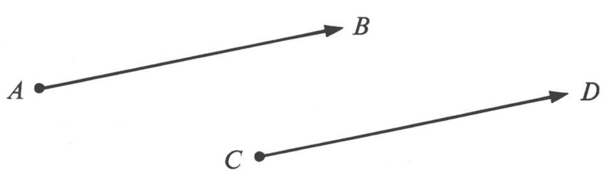

Figure 1.1

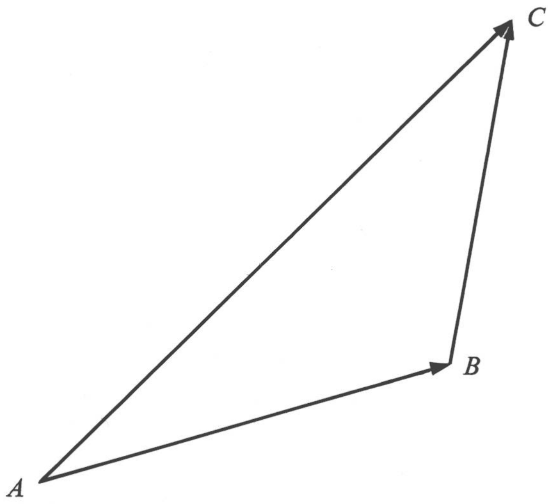

Figure 1.2

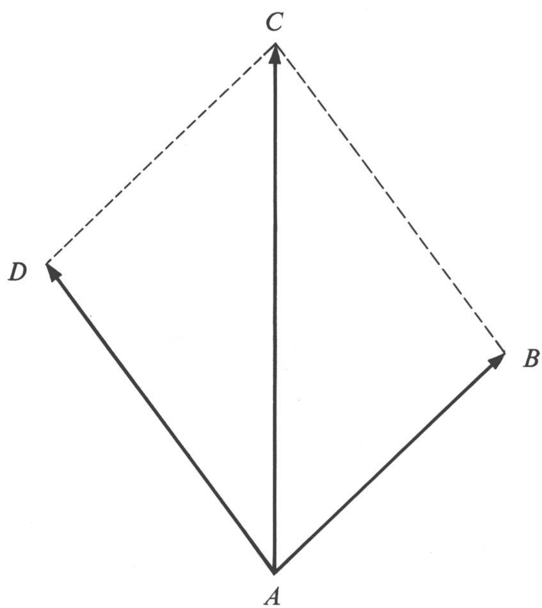

Figure 1.3

in Figs. 1.4 and 1.5.

That vector addition is commutative is illustrated in Fig. 1.6. Given vectors **AB** and **BC**, we select point D so that **AB** = **CD**. Then $ABCD$ is a parallelogram and

$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC}, \quad \mathbf{BC} + \mathbf{AB} = \mathbf{BC} + \mathbf{CD} = \mathbf{BD}.$$

Since **AC** = **BD**, we have proved the *commutative law:*

1.5 $$\mathbf{AB} + \mathbf{BC} = \mathbf{BC} + \mathbf{AB} \text{ for all } \mathbf{AB}, \mathbf{BC} \in V_2.$$

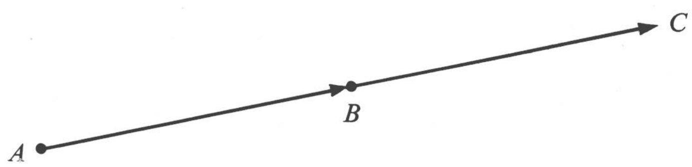

Figure 1.4

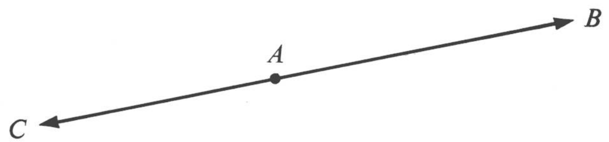

Figure 1.5

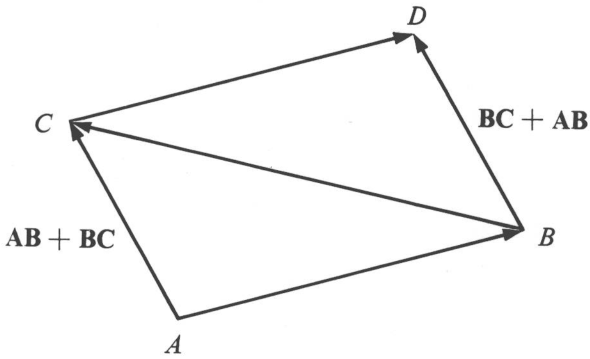

Figure 1.6