

9560231

科技资料

# Iterative Methods in Linear Algebra





0151-2-53

188

1991

9560281

# Iterative Methods in Linear Algebra

Proceedings of the IMACS International Symposium on  
Iterative Methods in Linear Algebra  
Brussels, Belgium, 2-4 April, 1991

edited by

**R. BEAUWENS**

*Service de Métrologie Nucléaire  
Université Libre de Bruxelles  
Brussels, Belgium*

**P. de GROEN**

*Department of Mathematics  
and Computer Science  
Vrije Universiteit Brussel  
Brussels, Belgium*



1992



E9560281

**NORTH-HOLLAND**  
**AMSTERDAM • LONDON • NEW YORK • TOKYO**

ELSEVIER SCIENCE PUBLISHERS B.V.  
Sara Burgerhartstraat 25  
P.O. Box 211, 1000 AE Amsterdam, The Netherlands

Distributors for the United States and Canada:

ELSEVIER SCIENCE PUBLISHING COMPANY INC.  
655 Avenue of the Americas  
New York, N.Y. 10010, U.S.A.

Library of Congress Cataloging-in-Publication Data

IMACS International Symposium on Iterative Methods in Linear Algebra  
(1991 : Brussels, Belgium)

Iterative methods in linear algebra : proceedings of the IMACS  
International Symposium on Iterative Methods in Linear Algebra,  
Brussels, Belgium, 2-4 April, 1991 / edited by R. Beauwens, P. de  
Groen.

1. cm.

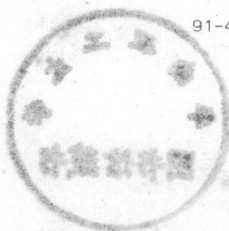
A symposium organized jointly by the International Association for  
Mathematics and Computers in Simulation and by the Free Universities  
of Brussels.

Includes index.

ISBN 0-444-89248-6

1. Algebras, Linear--Congresses. 2. Iterative methods  
(Mathematics)--Congresses. I. Beauwens, R. (Robert), 1939-  
II. De Groen, P. (Pieter), 1944- III. International Association  
for Mathematics and Computers in Simulation. IV. Vrije Universiteit  
Brussel. V. Université libre de Bruxelles. VI. Title.  
QA184.I44 1991  
512'.5--dc20

91-44923  
CIP



ISBN: 0 444 89248 6

© 1992 IMACS. All rights reserved.

*No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior written permission of the copyright owner, IMACS (IMACS Secretariat, Department of Computer Science, Rutgers University, New Brunswick, NJ 08903, U.S.A.)*

*Special regulations for readers in the U.S.A. - This publication has been registered with the Copyright Clearance Center Inc. (CCC), Salem, Massachusetts. Information can be obtained from the CCC about conditions under which photocopies of parts of this publication may be made in the U.S.A. All other copyright questions, including photocopying outside of the U.S.A., should be referred to the copyright owner, unless otherwise specified.*

*No responsibility is assumed by the publisher or by IMACS for any injury and/or damage to persons or property as a matter of products liability, negligence or otherwise, or from any use or operation of any methods, products, instructions or ideas contained in the material herein.*

pp. 459-468, 603-612: Copyright not transferred.

PRINTED IN THE NETHERLANDS

## PREFACE

The IMACS International Symposium on Iterative Methods in Linear Algebra, organized jointly by the International Association for Mathematics and Computers in Simulation (IMACS) and by the Free Universities of Brussels (ULB<sup>1</sup> and VUB<sup>2</sup>), took place in the Aula of the VUB from April 2 to April 4, 1991. It gathered more than 100 participants from more than 25 countries. Among them we could welcome a significant delegation from eastern Europe.

The purpose of the symposium was to provide a forum for the presentation and the discussion of recent advances in the analysis and implementation of iterative methods for solving large linear systems of equations and for computing eigenvalues, eigenvectors or singular values of large matrices. The contributions covered a broad range of subjects. The spectrum varied from detailed analyses of the multiple facets of the conjugate gradient method and of its various extensions, to new insights, novel applications and more speculative areas on one side, and to the technical aspects of parallel and vector implementations and of software development on the other.

There were seven invited plenary lectures. A. van der Sluis opened the symposium with a deep analysis of the convergence behaviour of conjugate gradients and Ritz values. O. Axelsson followed, disclosing the algorithms, devised with P. Vassilevski, for the construction of variable step preconditioners. Later in the first day afternoon, A. Yereimin explained how to construct sparse approximate inverse preconditioners for solving the 3D Navier-Stokes equations by GMRES on massively parallel computers.

H. van der Vorst opened the second day meeting with a wide overview of conjugate gradient type methods for nonsymmetric systems. In that afternoon, D. Kincaid reported on his joint work with D. Young on stationary second degree methods. By the end of the second day, S. Doi excited the audience by a video animation of the unexpected behaviour of the error evolution in the various schemes, presented in the morning session.

The third day meeting was opened by E. Wachspress (who incidentally confessed that he had been very much interested in SOR when he was ... young) with a status report on consistent sparse factorization, jointly developed with W. Noronha. F. Chatelin had the perilous job of closing the conference, which she did by presenting the elements of a new condition theory for the analysis of numerical algorithms. Thanks to her, nobody left before the very last minute.

Among the submitted contributions about 70 have been accepted for presentation at the symposium. From the variety of their titles, as displayed in the Table of Contents, one may infer the broadness of the subjects covered. Nevertheless, the dominant themes are the development of parallel and vector algorithms, the methods devised for solving unsymmetric problems and the PCG solution of symmetric problems. Most of these contributions are fully documented in these proceedings. A few ones are absent or reduced to summaries because of time delays,

<sup>1</sup> Université Libre de Bruxelles, 50 Av. F.D. Roosevelt, B - 1050 Bruxelles.

<sup>2</sup> Vrije Universiteit Brussel, Pleinlaan 2, B - 1050 Brussel.

publication elsewhere and other constraints, or did not survive the refereeing. In particular, the important contribution of J. Halton on the Monte Carlo solution of linear systems, for which we reserved the whole session 15, could not be included here, due to lack of time and space. In this session, J. Halton presented a condensed version of a monograph in preparation which is awaiting completion and publication in the near future. Requests for preliminary copies may be addressed to the author<sup>3</sup>.

Contributed papers have been both solicited and refereed by the members of the International Program Committee listed on page vii and we heartily thank them for the hard job they did. Special thanks are addressed here to O. Axelsson, M. Deville, M. Eiermann, D. Kincaid, G. Latouche, E. Mund, W. Niethammer, H. van der Vorst, E. Wachspress and Ch. Wu, who spent lots of time to organize special sessions. However, the International Program Committee alone would not have been able to do all the reviewing work. More than 100 contributions have been submitted, each of which had to be reviewed by two referees. We called the help of many other referees and it is our pleasure to acknowledge here their kind and quick anonymous assistance.

All contributions submitted for these proceedings had to be composed under  $\text{\LaTeX}$  format. We thank the authors for their compliance to this requirement, which enabled us to produce at the meeting a first version of the proceedings, containing most papers, namely all those prepared in the required (SIAM-) style. We are greatly indebted in this respect to the technical assistance of Dr. J.-C. Dehaes who so kindly put his deep knowledge of  $\text{\LaTeX}$  to our service and of Mrs J. Immers who reformatted a great number of papers.

Last but not least, symposia can hardly be organized without financial help and we heartily thank here the Belgian "Fonds National de la Recherche Scientifique / Nationaal Fonds voor Wetenschappelijk Onderzoek", "l'Exécutif de la Communauté Française de Belgique", IBM Belgium and Honeywell Europe S.A. for their assistance. Finally we thank the VUB for putting to our disposition their lecture hall and its technical staff, and we thank Linda Dasseville for her accurate secretarial work.

Pieter de Groen

October 1991

Robert Beauwens

<sup>3</sup> Department of Computer Science, The University of North Carolina, Chapel Hill, NC 27599-3175, USA.  
Email: halton@cs.unc.edu



## International Program Committee

O. Axelsson	KUN, Nijmegen, The Netherlands
R.E. Bank	UCSD, San Diego, USA
T.F. Chan	UCLA, Los Angeles, USA
F. Chatelin	IBM, Paris, France
P.J. Courtois	PRL, Louvain-la-Neuve, Belgium
M. Crouzeix	URI, Rennes, France
M. Deville	UCL, Louvain-la-Neuve, Belgium
E. Dick	RUG, Gent, Belgium
I.S. Duff	RAL, Chilton, UK
J.J. Dongarra	UT, Knoxville, USA
M. Eiermann	UK, Karlsruhe, W. Germany
S.C. Eisenstat	YU, Yale, USA
L. Elsner	UB, Bielefeld, W. Germany
D.J. Evans	LUT, Loughborough, UK
A. Hadjidimos	PU, Purdue, USA
S. Hammarling	NAG, Oxford, UK
P. Hemker	CWI, Amsterdam, The Netherlands
J.P. Hennart	UNAM, Mexico
Ch. Hirsch	VUB, Brussels, Belgium
D.R. Kincaid	UTA, Austin, USA
G. Latouche	ULB, Brussels, Belgium
J.F. Maître	ECL, Lyon, France
J. Mandel	UC, Denver, USA
E. Mund	UCL, Louvain-la-Neuve, Belgium
W. Niethammer	UK, Karlsruhe, W. Germany
S.V. Parter	UWM, Madison, USA
W. Queck	TUC, Chemnitz, E. Germany
A. Ruhe	UG, Göteborg, Sweden
Y. Saad	RIACS, Moffet Field, USA
Ph. Toint	FUNDP, Namur, Belgium
H.A. van der Vorst	RUU, Utrecht, The Netherlands
P. Van Dooren	PRL, Louvain-la-Neuve, Belgium
R.S. Varga	KSU, Kent, USA
R. Verfürth	UZ, Zürich, Switzerland
E.L. Wachspress	UT, Knoxville, USA
C.H. Wu	UL, Linz, Austria
H. Yserentant	UT, Tübingen, W. Germany

## TABLE OF CONTENTS

<b>INVITED LECTURES</b> .....	1
O. Axelsson and P.S. Vassilevski, <i>Construction of variable-step preconditioners for inner-outer iteration methods</i> .....	1
Françoise Chatelin and Valérie Frayssé, <i>Elements of a condition theory for the computational analysis of algorithms</i> .....	15
David R. Kincaid and David M. Young, <i>Stationary second-degree iterative methods and recurrences</i> .....	27
A. van der Sluis, <i>The convergence behaviour of conjugate gradients and Ritz values in various circumstances</i> .....	49
H.A. van der Vorst, <i>Conjugate gradient type methods for nonsymmetric linear systems</i> .....	67
W.P. Noronha and E. L. Wachspress <i>Consistent sparse factorization</i> .....	77
Ju.B. Lifshitz, A.A. Nikishin and A.Yu. Yeremin, <i>Sparse approximate inverse preconditionings for solving 3D CFD problems on massively parallel computers</i> .....	83
<b>Session 1: Coupled inner outer iteration methods</b> .....	85
Andrey V. Knyazev, <i>Iterative solution of PDE with strongly varying coefficients: algebraic version</i> .....	85
G. Haase, U. Langer and A. Meyer, <i>The Dirichlet domain decomposition method using inner multigrid solvers</i> .....	91
Junping Wang, <i>Convergence analysis of Schwarz algorithm and multilevel decomposition iterative methods I: selfadjoint and positive definite elliptic problems</i> .....	93
Alvaro L.G.A. Coutinho, José L.D. Alves and Nelson F.F. Ebecken, <i>Two-level preconditioners for hierarchical finite element equations and their applications in solid mechanics</i> .....	111
<b>Session 2: Numerical methods for the analysis of Markov models</b> .....	121
M. Coevoet, <i>Nested iteration for fractal images using the associated Markov process</i> .....	121
François Bonhoure, Yves Dallery and William J. Stewart, <i>On the numerical solution of Markov chains with periodic graphs</i> .....	131
Isi Mitrani and Debasis Mitra, <i>A spectral expansion method for random walks on semi-infinite strips</i> .....	141

<b>Session 3: Iterative solution of unsymmetric systems</b>	151
Roland W. Freund and Noël M. Nachtigal,	
<i>QMR: a quasi-minimal residual method for non-Hermitian linear systems</i>	151
C. Vuik,	
<i>A comparison of some GMRES-like methods</i>	155
Carsten Gellrich,	
<i>Generalized cg-methods with scaling for solving linear systems with M-matrices</i>	163
Auke van der Ploeg,	
<i>Preconditioning techniques for large sparse, non symmetric matrices with arbitrary sparsity patterns</i>	173
<b>Session 4: Spectral methods</b>	181
M.O. Deville and E.H. Mund,	
<i>Finite element preconditioning of collocation schemes for advection diffusion equations</i>	181
Y. Maday, R. Muñoz, A. Patera and E. Rønquist	
<i>Spectral element multigrid methods</i>	191
W. Heinrichs,	
<i>Finite element versus finite difference preconditioning for spectral multigrid methods</i>	203
S. Bertoluzza and D. Funaro,	
<i>Remarks about the matrices relative to the pseudospectral approximation of Neumann problems</i>	209
<b>Session 5: The Lyapunov equation</b>	217
A. Scottedward Hodel,	
<i>Recent applications of the Lyapunov equation in control theory</i>	217
Eugene L. Wachspress,	
<i>ADI Iterative solution of Lyapunov equations</i>	229
Gerhard Starke,	
<i>SOR-like methods for Lyapunov matrix equations</i>	233
<b>Session 6: Parallel and vector iterative methods</b>	241
M. Arioli, I.S. Duff, D. Ruiz and M. Sadkane,	
<i>Block Lanczos techniques for accelerating the block Cimmino method</i>	241
David J. Evans and Changjun Li,	
<i>The alternating group explicit (AGE) iterative method and its parallel implementation</i>	243
Ilan Bar-On,	
<i>New divide and conquer parallel algorithms for the Cholesky decomposition and Gram-Schmidt process</i>	251
Michael Griebel, Michael Schneider and Christoph Zenger,	
<i>A combination technique for the solution of sparse grid problems</i>	263
Rüdiger Weiss and Willi Schönauer,	
<i>Accelerating generalized conjugate gradient methods by smoothing</i>	283
Hans-Joachim Bungartz,	
<i>An adaptive poisson solver using hierarchical bases and sparse grids</i>	293



L.Yu. Kolotilina, A.A. Nikishin and A.Yu. Yeremin, <i>Factorized sparse approximate inverse (FSAI) preconditionings for solving 3D FE systems on massively parallel computers. II. Iterative construction of FSAI preconditioners</i> .....	311
Rod Cook, <i>A reformulation of preconditioned conjugate gradients suitable for a local memory multi-processor</i> .....	313
Malathi Ramdas and David R. Kincaid, <i>Parallelizing ITPACKV 2D for the Cray Y-MP</i> .....	323
E.M. Daoudi and P. Manneback, <i>Implementation of ICCG algorithm on distributed memory architecture</i> .....	339
Seiji Fujino and Shun Doi, <i>Optimizing Multicolor ICCG Methods on some Vectorcomputers</i> .....	349
<b>Session 7: Complex variable methods</b> .....	359
Michael Eiermann, <i>The numerical radius of certain SOR iteration matrices</i> .....	359
Carl Jagels and Lothar Reichel, <i>The isometric Arnoldi process and an application to iterative solution of large linear systems</i> .....	361
Gerhard Starke, <i>A generalization of Faber polynomials for the ADI method</i> .....	371
Martin H. Gutknecht, <i>On certain types of <math>(k, l)</math>-step methods for solving linear systems of equations</i> ....	373
<b>Session 8: Basic iterative methods</b> .....	381
A. Haegemans and J. Verbeke, <i>The GAOR-method in the complex case</i> .....	381
Ludwig Elsner and Volker Mehrmann, <i>Convergence of block iterative methods for matrices arising in fluid flow computations</i> .....	391
M. Legua, L. Jódar and A.G. Law, <i>Existence of numerical solutions for singular partial differential systems</i> .....	395
Alfredo N. Iusem, <i>On the convergence of iterative methods for symmetric linear complementarity problems</i> .....	401
Sabine Van Huffel, <i>Iterative methods for solving total least squares problems</i> .....	403
<b>Session 9: Eigenvalue problems</b> .....	415
Mario Ahues and Alain Largillier, <i>Rayleigh-Schrödinger series versus inexact Newton methods for spectral computations</i> .....	415
Peter Sternecker, Lutz Gross and Willi Schönauer, <i>A polyalgorithm for the solution of large symmetric general eigenproblems</i> .....	423
Miloslav Znojil, <i>Acceleration of convergence of the Hill-determinant calculations</i> .....	433

Nahid Emad and Serge Petiton, <i>Numerical behaviour of iterative Arnoldi's method for sparse eigenproblems on massively parallel architectures</i> .....	441
<b>Session 10: Methods for nonsymmetric systems</b> .....	445
C. Brezinski, M. Redivo Zaglia and H. Sadok, <i>A block bordering method for the treatment of breakdown in the biconjugate gradient algorithm</i> .....	445
Artur Walter, <i>Sparse secant methods for the iterative solution of large nonsymmetric linear systems</i>	449
J.J. Rusch, <i>Using a complex version of GMRES for solving optical scattering problems</i> .....	459
David J. Evans and Changjun Li, <i>A note on working with the reduced linear system</i> .....	469
<b>Session 11: Mixed hybrid methods</b> .....	475
Bruno Despres, Patrick Joly and Jean E. Roberts, <i>A domain decomposition method for the harmonic Maxwell equations</i> .....	475
J.P. Hennart, <i>A mixed-hybrid finite element formulation of fast nodal elliptic solvers</i> .....	485
<b>Session 12: Semiconductor device equations</b> .....	493
J. Fuhrmann and K. Gärtner, <i>Incomplete factorization and linear multigrid algorithms for the semiconductor device equations</i> .....	493
Arnold Reusken, <i>Multigrid applied to two-dimensional exponential fitting for drift-diffusion models</i>	505
<b>Session 13: Preconditioned conjugate gradients</b> .....	515
J.J. Júdice J.M. Patricio, <i>A Truncated envelope preconditioning technique</i> .....	515
Valery P. Il'in, <i>The analysis of some explicit and implicit incomplete factorization methods</i> .....	517
Magolu Monga Made, <i>Sparse block approximate factorizations for singular problems</i> .....	519
A.J. Wathen, <i>Singular element preconditioning for the finite element method</i> .....	531
Sergey A. Sander, <i>Domain decomposition and rational approximation problems</i> .....	541
Y. Notay, <i>Upper eigenvalue bounds and related modified incomplete factorization strategies</i> .	551
Thomas Huckle, <i>Circulant/skewcirculant matrices as preconditioners for Hermitian Toeplitz systems</i>	563
Shlomo Shlafman and Ilan Efrat, <i>Using Korn's inequality for an efficient iterative solution of structural analysis problems</i> .....	575

Victor Eijkhout, <i>Beware of unperturbed modified incomplete factorizations</i> .....	583
Alison Ramage, <i>Eigenvalue clustering and conjugate gradient convergence for elliptic partial     differential equations</i> .....	593
E. F. D'Azevedo, P. A. Forsyth and Wei-Pai Tang, <i>Two variants of minimum discarded fill ordering</i> .....	603
<b>Session 14: Software developments</b> .....	613
Udo R. Krieger and Michael Sczittnick, <i>Application of numerical solution methods for singular systems in the field of     computational probability theory</i> .....	613
Werner Queck, <i>FEMGPL – A software package for solving elliptic boundary value problems on     personal computers</i> .....	627
Index of authors .....	635



## INVITED LECTURES

### CONSTRUCTION OF VARIABLE-STEP PRECONDITIONERS FOR INNER-OUTER ITERATION METHODS

O. AXELSSON\* AND P.S. VASSILEVSKI†

**Abstract.** The generalized conjugate gradient, GCG-method as proposed by the first author was recently applied by the present authors for the case of variable-step preconditioner, i.e. in general a nonlinear preconditioning mapping and its convergence was analyzed.

In a number of practical applications there arise naturally linear algebraic problems with matrices  $A$  partitioned in a two-by-two block form  $A = (A_{ij})_{i,j=1}^2$ .  $A$  can be indefinite and even nonsymmetric. For such matrices, a general framework for the construction of variable-step preconditioners utilizing parameter free inexact solvers, for instance by use of a conjugate gradient method to compute inner iterations for the first matrix block  $A_{11}$  and for the Schur complement matrix  $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ , was studied.

The disadvantage with this method is that the action of  $A_{11}$  is required two or more times during each outer iteration. These actions must be sufficiently accurate otherwise the rate of convergence can be too slow or even divergence can occur.

In the present paper a modified version of the algorithm is presented where only one accurate action is required per step in addition to an action required only for the computation of the length of a steepest descent step, which can therefore be less accurate.

The efficacy of this method will be evident in particular for problems arising in domain decomposition methods.

**Key words.** inner-outer iterations, variable-step preconditioner, GCG-method, domain decomposition, inexact solver.

**AMS(MOS) subject classifications.** 65F10, 65N30.

**1. Introduction.** We consider matrices partitioned in two-by-two block form

$$(1.1) \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where  $A$  and  $A_{11}$  are assumed nonsingular and possibly nonsymmetric and indefinite.

Based on the block-matrix factorization

$$A = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

where  $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ , the following familiar block-matrix form of  $A^{-1}$  is derived,

$$A^{-1} = \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$$

\* Faculty of Mathematics and Informatics, Catholic University, NL-6525 ED Nijmegen, The Netherlands.

† Department of Mathematics, University of Wyoming, University Station, P.O. Box 3036, Laramie, WY 82071, USA. Email: panayot@uwyo.bitnet

$$= \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{bmatrix}.$$

Then for a corresponding partitioning of a vector  $v$ ,  $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ , we have

$$(1.2) \quad A^{-1}v = \begin{bmatrix} -A_{11}^{-1}(A_{12}S^{-1}(v_2 - A_{21}A_{11}^{-1}(v_1))) + A_{11}^{-1}(v_1) \\ S^{-1}(v_2 - A_{21}A_{11}^{-1}(v_1)) \end{bmatrix}.$$

To compute this we need two actions of  $A_{11}^{-1}$ , or even three or more depending on the implementation, because each action of  $S$  involves among other things an action of  $A_{11}^{-1}$ .

As has been shown by the authors in [5], a variable-step preconditioner, i.e. a preconditioner which can change from one iteration to the next can be constructed using approximate solvers for  $A_{11}$  and  $S$  based on iterative methods using the conjugate gradient method for instance.

There are then given two, in general nonlinear mappings,

$$v_1 \rightarrow B_{11}[v_1], \quad v_2 \rightarrow C[v_2]$$

that satisfy

$$(1.3a) \quad \|A_{11}B_{11}[v_1] - v_1\|_0 \leq \varepsilon_1 \|v_1\|_0, \quad \text{for all } v_1$$

$$(1.3b) \quad \|SC[v_2] - v_2\|_0 \leq \varepsilon_2 \|v_2\|_0, \quad \text{for all } v_2$$

where  $\varepsilon_1, \varepsilon_2$  are sufficiently small positive numbers and norms,

$$\|v_1\|_0 = \{(\tilde{v}_1, \tilde{v}_1)_0\}^{\frac{1}{2}}, \quad \|v_2\|_0 = \{(\tilde{v}_2, \tilde{v}_2)_0\}^{\frac{1}{2}}$$

based on a corresponding inner product,

$$(v, v)_0 = (\tilde{v}_1, \tilde{v}_1)_0 + (\tilde{v}_2, \tilde{v}_2)_0,$$

where  $\tilde{v}_1 = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ ,  $\tilde{v}_2 = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$ . (For simplicity, we use the same notations for the norms in the different vector spaces.)

Every action of  $A_{11}^{-1}$  and  $S^{-1}$  in (1.2) is now replaced by their approximations  $B_{11}[\cdot]$  and  $C[\cdot]$ , respectively and this defines a variable-step (or nonlinear) preconditioner  $B[v]$  for  $A^{-1}v$ . The variable-step preconditioner proposed in [5] is then defined by the following algorithm:

#### ALGORITHM 1

- 1)  $w_1 = B_{11}[v_1]$
- 2)  $w_2 = -A_{21}w_1 + v_2$
- 3)  $x_2 = C[w_2]$
- 4)  $y_1 = A_{12}x_2$
- 5)  $z_1 = B_{11}[y_1]$
- 6)  $x_1 = w_1 - z_1$ .

Then  $B[v] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ . Note that if in Algorithm 1 we have  $B_{11}[v_1] = A_{11}^{-1}v_1$  and  $C[v_2] = S^{-1}v_2$ , we obtain  $B[v] = A^{-1}v$ .

The mappings  $B_{11}[\cdot]$  and  $C[\cdot]$  correspond in practice frequently to some inner iteration methods. To check if we have performed a sufficient number of such inner iterations, i.e. to check if the corresponding mappings  $B_{11}[\cdot]$  and  $C[\cdot]$  are sufficiently accurate, we can use the following tests, which involve only vectors computed during the iterations:

$$(1.4a) \quad \|A_{11}w_1 - v_1\|_0 \leq \varepsilon_1 \|v_1\|_0, \quad \|A_{11}z_1 - y_1\|_0 \leq \varepsilon_1 \|y_1\|_0,$$

which check if the iterative solutions  $w_1$  and  $z_1$  in steps 1) and 5), respectively are sufficiently accurate, and

$$(1.4b) \quad \|A_{22}x_2 - A_{21}z_1 - w_2\|_0 \leq \varepsilon_2 \|w_2\|_0$$

where  $w_2 = v_2 - A_{21}w_1$ , which checks if the solution  $x_2$  in step 3) is sufficiently accurate.

However, the last check involves the vector  $z_1$  which is available only in step 5). Hence (1.4b) is actually performed after step 5). This means that we may have to repeat steps 4) and 5) after we computed a more accurate solution in step 3), if (1.4b) failed to be satisfied initially.

In practice, it can be advisable to test instead on the sign of the leading coefficient in the conjugate gradient method (which must be positive if the preconditioned operator has a positive definite symmetric part, see [1]). If the sign test is violated, we repeat algorithm 1 with more inner iterations, which corresponds to choosing smaller values of  $\varepsilon_1, \varepsilon_2$ .

To estimate the deviation of  $AB[v]$  from  $v$  we note first that Algorithm 1 shows that

$$(1.5) \quad \begin{aligned} AB[v] - v &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} A_{11}(w_1 - z_1) + A_{12}x_2 \\ A_{21}(w_1 - z_1) + A_{22}x_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} A_{11}w_1 - v_1 - (A_{11}z_1 - y_1) \\ A_{22}x_2 - w_2 - A_{21}z_1 \end{bmatrix} \end{aligned}$$

Let

$$(1.6) \quad \sigma_1 = \|A_{12}S^{-1}\|_0, \quad \sigma_2 = \|A_{21}A_{11}^{-1}\|_0.$$

We assume that  $\varepsilon_1\sigma_2 < 1$ . Note now that (1.4a) shows that

$$(1.7) \quad \begin{aligned} \|w_2\|_0 &\leq \|v_2\|_0 + \|A_{21}A_{11}^{-1}A_{11}w_1\|_0 \leq \|v_2\|_0 + \sigma_2\|A_{11}w_1\|_0 \\ &\leq \|v_2\|_0 + \sigma_2(1+\varepsilon_1)\|v_1\|_0. \end{aligned}$$

Further

$$(1.8) \quad \begin{aligned} \|y_1\|_0 = \|A_{12}x_2\|_0 &= \|A_{12}S^{-1}SC[w_2]\|_0 \leq \sigma_1\|SC[w_2]\|_0 \\ &\leq \sigma_1[\|w_2\|_0 + \|SC[w_2] - w_2\|_0]. \end{aligned}$$

Now using (1.4 a,b) we get

$$\begin{aligned} \|SC[w_2] - w_2\|_0 &= \|Sx_2 - w_2\|_0 = \\ &= \|A_{22}x_2 - A_{21}A_{11}^{-1}A_{12}x_2 - w_2\|_0 = \\ &= \|A_{22}x_2 - A_{21}B_{11}[y_1] - w_2 + A_{21}(B_{11}[y_1] - A_{11}^{-1}A_{12}x_2)\|_0 \leq \\ &= \|A_{22}x_2 - A_{21}z_1 - w_2 + A_{21}A_{11}^{-1}[A_{11}z_1 - y_1]\|_0 \leq \\ &\leq \varepsilon_2\|w_2\|_0 + \sigma_2\varepsilon_1\|y_1\|_0. \end{aligned}$$



This, and (1.8) show that

$$\|y_1\|_0 \leq \sigma_1[\|w_2\|_0 + \varepsilon_2\|w_2\|_0]/(1-\varepsilon_1\sigma_2)$$

so, together with (1.7) we get

$$(1.9) \quad (1+\varepsilon_1)\|v_1\|_0]/(1-\varepsilon_1\sigma_2).$$

(1.5), (1.7), (1.4a,b) and (1.9) show now that

$$\begin{aligned} \|AB[v]-v\|_0 &\leq \{(\|A_{11}w_1-v_1\|_0 + \|A_{11}z_1-y_1\|_0)^2 + \|A_{22}x_2-w_2-A_{21}z_1\|_0^2\}^{\frac{1}{2}} \\ &\leq \|A_{11}w_1-v_1\|_0 + \|A_{11}z_1-y_1\|_0 + \|A_{22}x_2-w_2-A_{21}z_1\|_0 \\ &\leq \varepsilon_1\|v_1\|_0 + \varepsilon_1(1+\varepsilon_2)\sigma_1[\|v_2\|_0 + \sigma_2(1+\varepsilon_1)\|v_1\|_0]/(1-\varepsilon_1\sigma_2) \\ &\quad + \varepsilon_2\|v_2\|_0 + \varepsilon_2(1+\varepsilon_1)\sigma_2\|v_1\|_0 \\ &= [\varepsilon_1 + \varepsilon_1(1+\varepsilon_1)(1+\varepsilon_2)\sigma_1\sigma_2/(1-\varepsilon_1\sigma_2) + \varepsilon_2(1+\varepsilon_1)\sigma_2]\|v_1\|_0 \\ &\quad + [\varepsilon_2 + \varepsilon_1(1+\varepsilon_2)\sigma_1/(1-\varepsilon_1\sigma_2)]\|v_2\|_0 \\ &\leq C(\varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2)\|v\|_0, \end{aligned}$$

where  $\|v\|_0 = \{\|v_1\|^2 + \|v_2\|^2\}^{\frac{1}{2}}$ . Here

$$(1.10) \quad C(\varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2) \leq \sqrt{2} \max\{\varepsilon_1 + \varepsilon_1(1+\varepsilon_1)(1+\varepsilon_2)\sigma_1\sigma_2/(1-\varepsilon_1\sigma_2) + \varepsilon_2(1+\varepsilon_1)\sigma_2, \\ [\varepsilon_2 + \varepsilon_1(1+\varepsilon_2)\sigma_1/(1-\varepsilon_1\sigma_2)]\}.$$

Note that

$$C(\varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2) < \sqrt{2}[\varepsilon_1 + \varepsilon_1\sigma_1 + \varepsilon_1\sigma_1\sigma_2 + \varepsilon_2 + \varepsilon_2\sigma_2], \quad \varepsilon_1, \varepsilon_2 \rightarrow 0.$$

In [5] it was shown that for  $\varepsilon_1, \varepsilon_2$  sufficiently small,  $AB[\cdot]$  is coercive and bounded, namely

$$(v, AB[v])_0 \geq (1-C^{\frac{1}{2}})\|v\|_0^2, \quad \text{for all } v$$

and

$$\|AB[v]\|_0 \leq (1+C^{\frac{1}{2}})\|v\|_0^2, \quad \text{for all } v.$$

It was also shown that for the residuals  $r^k = Ax^k - b$ ,  $k = 0, 1, \dots$  computed by a preconditioned generalized conjugate gradient (GCG) method, or even by a preconditioned steepest descent method, with the variable-step preconditioner  $B[\cdot]$ , the following convergence rate estimate holds:

$$\|r^k\|_0 \leq \left(1 - \left(\frac{\delta_1}{\delta_2}\right)^2\right)^{\frac{k}{2}} \|r^0\|_0, \quad k \geq 0,$$

where  $\delta_1 = 1 - C^{\frac{1}{2}}$ ,  $\delta_2 = 1 + C^{\frac{1}{2}}$ .

Note that the above method is parameter free, i.e. for a user the method looks like a "black-box".

In Axelsson and Vassilevski [5] several particular applications of the construction of variable-step preconditioners are demonstrated, namely for two-level hierarchical iterative methods for

the finite element discretization of non-self adjoint elliptic problems, for mixed finite element discretization of second order elliptic problems and for Stokes equation. Related results for the latter application were studied earlier by Bank, Welfert and Yserentant [9], by Langer and Queck [17], Verfürth [22] and more recently with application to domain decomposition methods with inexact subdomain solvers by Börgers [11], Langer [16] and Y. Vasilevskij [21]. However, in all of these papers the constructed preconditioner is either a fixed matrix or the global (outer) iterative method is a stationary one, i.e. not of a variational type.

In the present paper we simplify our original method in such a way that only one accurate inner iteration step with the mapping  $B_{11}[\cdot]$  is required per outer iteration. This is made possible by use of a parameter free method to construct approximate inverses for the action of an approximate Schur complement  $\tilde{S}v_2 = A_{22}v_2 - A_{21}B_{11}[A_{12}v_2]$  or for the residuals arising from the approximate Schur complement reduced system for a defect corrected global system. Although we work on the reduced system we can show convergence for the global system and with a rate which depends only on the accuracy of the mapping  $B_{11}[\cdot]$  and of the approximate inverse of the Schur complement.

Before presenting the new method we consider a general method to construct approximate inverses to nonlinear mappings that are almost linear. This method will then be applied when we compute actions of approximations of the inverses of the Schur complement matrix.

**2. Construction of approximate inverses to nonlinear mappings that are almost linear.** Consider first the following nonlinear equation,

$$(2.1) \quad \tilde{A}[x] = v,$$

where the mapping  $\tilde{A}[\cdot]$  is assumed to be sufficiently close to a linear in the sense that for some matrix  $A$ , we have

$$(2.2) \quad \|Ax - \tilde{A}[x]\|_1 \leq \delta \|Ax\|_1, \quad \text{for all } x,$$

where  $\delta \in (0, 1)$  is sufficiently small and where  $\|x\|_1 = \sqrt{(x, x)_1}$  is a given norm, defined by an inner product  $(\cdot, \cdot)_1$ . The mappings  $A$  and  $\tilde{A}$  can be preconditioned forms of some operators  $\hat{A}$  and  $\tilde{\hat{A}}$ , say, by some preconditioning matrix  $D$ . For notational simplicity we do not give this preconditioner in explicit form. As an application of this, in the following section,  $A$  will be a Schur complement matrix and  $\tilde{A}$  an approximation of  $A$  whose closeness to  $A$  is fully controlled by the number of inner iterations used to solve systems with the top matrix block of the original global system.

We assume that  $A$  satisfies the following boundedness and coercivity properties, i.e. for some positive constants  $\gamma_1, \gamma_2$  we have

$$(2.3) \quad \|Ax\|_1 \leq \gamma_2 \|x\|_1, \quad \text{for all } x,$$

$$(2.4) \quad (Ax, x)_1 \geq \gamma_1 (x, x)_1, \quad \text{for all } x.$$

Note that for this to hold the matrix  $A$  need not be symmetric.

We consider now the following variational type algorithm to approximately solve the nonlinear equation (2.1).

**ALGORITHM 2.**

initiate: choose  $x^0$ ;  $r^0 = v - \tilde{A}[x^0]$ ;

for  $k = 1, 2, \dots$  compute

$$\tilde{r}^{k-1} = \tilde{A}[r^{k-1}];$$

$$\alpha_{k-1} = (r^{k-1}, \tilde{r}^{k-1})_1 / (\tilde{r}^{k-1}, \tilde{r}^{k-1})_1;$$

$$x^k = x^{k-1} + \alpha_{k-1} r^{k-1};$$

$$r^k = v - \tilde{A}[x^k];$$

It can be seen that for a linear mapping, i.e. a matrix  $\tilde{A}$ , the above algorithm reduces to the steepest descent method to compute iterations  $x^k$  for which we have monotone convergence with steepest descent method decay in the gradient of the functional  $(r^k, r^k)_1$ .

The next theorem shows that algorithm 2 gives an approximate solution of (2.1), which converges with a geometric rate of convergence.

**THEOREM 2.1.** *The  $k$ -th iterate  $x^k$ ,  $k \geq 1$ , generated by Algorithm 2 satisfies the inequality,*

$$\|v - \tilde{A}[x^k]\|_1 \leq \left[ q^k + \frac{1}{1-q} \frac{2\delta}{1-\delta} \right] \|v\|_1,$$

where

$$q = \sqrt{1 - (\tilde{\gamma}_1/\tilde{\gamma}_2)^2} + 4\delta/(1-\delta),$$

$\tilde{\gamma}_1 = \gamma_1 - \delta\gamma_2$ ,  $\tilde{\gamma}_2 = (1+\delta)\gamma_2$ ,  $\delta$  is the constant in (2.2) and where  $\gamma_1, \gamma_2$  are defined in (2.3), (2.4), respectively. We assume that  $\delta$ ,  $0 < \delta < 1$ , is sufficiently small to make  $q < 1$ , and we see that we can get an arbitrarily accurate solution  $x^k$  satisfying

$$\|v - \tilde{A}[x^k]\|_1 \leq [\varepsilon + O(\delta)] \|v\|_1$$

by choosing  $\delta$  sufficiently small and performing  $k = O(\log \frac{1}{\varepsilon})$  iterations, where  $\varepsilon$  is the relative stopping accuracy.

*Proof.* First we show that  $\tilde{A}[\cdot]$  itself is bounded and coercive. Using (2.2), (2.3) and (2.4) one readily derives

$$(\tilde{A}[x], x)_1 = (\tilde{A}[x] - Ax, x)_1 + (Ax, x)_1 \geq (\gamma_1 - \delta\gamma_2)(x, x)_1,$$

and

$$\|\tilde{A}[x]\|_1 \leq (1+\delta)\|Ax\|_1 \leq (1+\delta)\gamma_2\|x\|_1.$$

Hence, for the  $(k-1)$ st residual  $r^{k-1}$  in Algorithm 2, we have

$$(2.5) \quad \begin{aligned} (r^{k-1}, \tilde{A}[r^{k-1}])_1 &\geq \tilde{\gamma}_1 \|r^{k-1}\|_1^2 \\ (\tilde{A}[r^{k-1}], \tilde{A}[r^{k-1}])_1 &\leq \tilde{\gamma}_2^2 \|r^{k-1}\|_1^2, \end{aligned}$$