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Iterative Methods in Linear Algebra

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Proceedings of the IMACS International Symposium on Iterative Methods in Linear Algebra
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PREFACE

The IMACS International Symposium on Iterative Methods in Linear Algebra, organized jointly by the International Association for Mathematics and Computers in Simulation (IMACS) and by the Free Universities of Brussels (ULB[1] and VUB[2]), took place in the Aula of the VUB from April 2 to April 4, 1991. It gathered more than 100 participants from more than 25 countries. Among them we could welcome a significant delegation from eastern Europe.

The purpose of the symposium was to provide a forum for the presentation and the discussion of recent advances in the analysis and implementation of iterative methods for solving large linear systems of equations and for computing eigenvalues, eigenvectors or singular values of large matrices. The contributions covered a broad range of subjects. The spectrum varied from detailed analyses of the multiple facets of the conjugate gradient method and of its various extensions, to new insights, novel applications and more speculative areas on one side, and to the technical aspects of parallel and vector implementations and of software development on the other.

There were seven invited plenary lectures. A. van der Sluis opened the symposium with a deep analysis of the convergence behaviour of conjugate gradients and Ritz values. O. Axelsson followed, disclosing the algorithms, devised with P. Vassilevski, for the construction of variable step preconditioners. Later in the first day afternoon, A. Yeremin explained how to construct sparse approximate inverse preconditioners for solving the 3D Navier-Stokes equations by GMRES on massively parallel computers.

H. van der Vorst opened the second day meeting with a wide overview of conjugate gradient type methods for nonsymmetric systems. In that afternoon, D. Kincaid reported on his joint work with D. Young on stationary second degree methods. By the end of the second day, S. Doi excited the audience by a video animation of the unexpected behaviour of the error evolution in the various schemes, presented in the morning session.

The third day meeting was opened by E. Wachspress (who incidentally confessed that he had been very much interested in SOR when he was . . . young) with a status report on consistent sparse factorization, jointly developed with W. Noronha. F. Chatelin had the perilous job of closing the conference, which she did by presenting the elements of a new condition theory for the analysis of numerical algorithms. Thanks to her, nobody left before the very last minute.

Among the submitted contributions about 70 have been accepted for presentation at the symposium. From the variety of their titles, as displayed in the Table of Contents, one may infer the broadness of the subjects covered. Nevertheless, the dominant themes are the development of parallel and vector algorithms, the methods devised for solving unsymmetric problems and the PCG solution of symmetric problems. Most of these contributions are fully documented in these proceedings. A few ones are absent or reduced to summaries because of time delays,

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publication elsewhere and other constraints, or did not survive the refereeing. In particular, the important contribution of J. Halton on the Monte Carlo solution of linear systems, for which we reserved the whole session 15, could not be included here, due to lack of time and space. In this session, J. Halton presented a condensed version of a monograph in preparation which is awaiting completion and publication in the near future. Requests for preliminary copies may be addressed to the author[3].

Contributed papers have been both solicited and refereed by the members of the International Program Committee listed on page vii and we heartily thank them for the hard job they did. Special thanks are addressed here to O. Axelsson, M. Deville, M. Eiermann, D. Kincaid, G. Latouche, E. Mund, W. Niethammer, H. van der Vorst, E. Wachspress and Ch. Wu, who spent lots of time to organize special sessions. However, the International Program Committee alone would not have been able to do all the reviewing work. More than 100 contributions have been submitted, each of which had to be reviewed by two referees. We called the help of many other referees and it is our pleasure to acknowledge here their kind and quick anonymous assistance.

All contributions submitted for these proceedings had to be composed under LaTeX format. We thank the authors for their compliance to this requirement, which enabled us to produce at the meeting a first version of the proceedings, containing most papers, namely all those prepared in the required (SIAM-) style. We are greatly indebted in this respect to the technical assistance of Dr. J.-C. Dehaes who so kindly put his deep knowledge of LaTeX to our service and of Mrs J. Immers who reformatted a great number of papers.

Last but not least, symposia can hardly be organized without financial help and we heartily thank here the Belgian "Fonds National de la Recherche Scientifique / Nationaal Fonds voor Wetenschappelijk Onderzoek", "l'Exécutif de la Communauté Française de Belgique", IBM Belgium and Honeywell Europe S.A. for their assistance. Finally we thank the VUB for putting to our disposition their lecture hall and its technical staff, and we thank Linda Dasseville for her accurate secretarial work.

Pieter de Groen October 1991 Robert Beauwens

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International Program Committee

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INVITED LECTURES

CONSTRUCTION OF VARIABLE-STEP PRECONDITIONERS FOR INNER-OUTER ITERATION METHODS

O. AXELSSON* AND P.S. VASSILEVSKI†

Abstract. The generalized conjugate gradient, GCG-method as proposed by the first author was recently applied by the present authors for the case of variable-step preconditioner, i.e. in general a nonlinear preconditioning mapping and its convergence was analyzed.

In a number of practical applications there arise naturally linear algebraic problems with matrices A partitioned in a two-by-two block form $A = (A_{i,j})_{i,j=1}^2$. A can be indefinite and even nonsymmetric. For such matrices, a general framework for the construction of variable-step preconditioners utilizing parameter free inexact solvers, for instance by use of a conjugate gradient method to compute inner iterations for the first matrix block A_{11} and for the Schur complement matrix $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$, was studied.

The disadvantage with this method is that the action of A_{11} is required two or more times during each outer iteration. These actions must be sufficiently accurate otherwise the rate of convergence can be too slow or even divergence can occur.

In the present paper a modified version of the algorithm is presented where only one accurate action is required per step in addition to an action required only for the computation of the length of a steepest descent step, which can therefore be less accurate.

The efficacy of this method will be evident in particular for problems arising in domain decomposition methods.

Key words. inner-outer iterations, variable-step preconditioner, GCG-method, domain decomposition, inexact solver.

AMS(MOS) subject classifications. 65F10, 65N30.

1. Introduction. We consider matrices partitioned in two-by-two block form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.1}$$

where A and A_{11} are assumed nonsingular and possibly nonsymmetric and indefinite.

Based on the block-matrix factorization

$$A = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

where $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$, the following familiar block-matrix form of A^{-1} is derived,

$$A^{-1} = \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix}$$

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$$= \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{bmatrix}.$$

Then for a corresponding partitioning of a vector v, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, we have

$$A^{-1}v = \begin{bmatrix} -A_{11}^{-1}(A_{12}S^{-1}(v_2 - A_{21}A_{11}^{-1}(v_1))) + A_{11}^{-1}(v_1) \\ S^{-1}(v_2 - A_{21}A_{11}^{-1}(v_1)) \end{bmatrix}. \tag{1.2}$$

To compute this we need two actions of A_{11}^{-1}, or even three or more depending on the implementation, because each action of S involves among other things an action of A_{11}^{-1}.

As has been shown by the authors in [5], a variable-step preconditioner, i.e. a preconditioner which can change from one iteration to the next can be constructed using approximate solvers for A_{11} and S based on iterative methods using the conjugate gradient method for instance.

There are then given two, in general nonlinear mappings,

$$v_1 \to B_{11}[v_1], \quad v_2 \to C[v_2]$$

that satisfy

$$\|A_{11}B_{11}[v_1] - v_1\|_0 \le \varepsilon_1 \|v_1\|_0, \quad \text{for all } v_1 \tag{1.3a}$$

$$\|SC[v_2] - v_2\|_0 \le \varepsilon_2 \|v_2\|_0, \quad \text{for all } v_2 \tag{1.3b}$$

where $\varepsilon_1, \varepsilon_2$ are sufficiently small positive numbers and norms,

$$\|v_1\|_0 = \{(\tilde{v}_1, \tilde{v}_1)_0\}^{\frac{1}{2}}, \quad \|v_2\|_0 = \{(\tilde{v}_2, \tilde{v}_2)_0\}^{\frac{1}{2}}$$

based on a corresponding inner product,

$$(v, v)_0 = (\tilde{v}_1, \tilde{v}_1)_0 + (\tilde{v}_2, \tilde{v}_2)_0,$$

where $\tilde{v}_1 = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$, $\tilde{v}_2 = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$. (For simplicity, we use the same notations for the norms in the different vectorspaces.)

Every action of A_{11}^{-1} and S^{-1} in (1.2) is now replaced by their approximations $B_{11}[\cdot]$ and $C[\cdot]$, respectively and this defines a variable-step (or nonlinear) preconditioner $B[v]$ for $A^{-1}v$. The variable-step preconditioner proposed in [5] is then defined by the following algorithm:

ALGORITHM 1
1) $w_1 = B_{11}[v_1]$
2) $w_2 = -A_{21}w_1 + v_2$
3) $x_2 = C[w_2]$
4) $y_1 = A_{12}x_2$
5) $z_1 = B_{11}[y_1]$
6) $x_1 = w_1 - z_1$.

Then $B[v] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Note that if in Algorithm 1 we have $B_{11}[v_1] = A_{11}^{-1}v_1$ and $C[v_2] = S^{-1}v_2$, we obtain $B[v] = A^{-1}v$.

The mappings $B_{11}[\cdot]$ and $C[\cdot]$ correspond in practice frequently to some inner iteration methods. To check if we have performed a sufficient number of such inner iterations, i.e. to check if the corresponding mappings $B_{11}[\cdot]$ and $C[\cdot]$ are sufficiently accurate, we can use the following tests, *which involve only vectors computed during the iterations:*

$$\|A_{11}w_1 - v_1\|_0 \le \varepsilon_1\|v_1\|_0, \quad \|A_{11}z_1 - y_1\|_0 \le \varepsilon_1\|y_1\|_0, \tag{1.4a}$$

which check if the iterative solutions w_1 and z_1 in steps 1) and 5), respectively are sufficiently accurate, and

$$\|A_{22}x_2 - A_{21}z_1 - w_2\|_0 \le \varepsilon_2\|w_2\|_0 \tag{1.4b}$$

where $w_2 = v_2 - A_{21}w_1$, which checks if the solution x_2 in step 3) is sufficiently accurate.

However, the last check involves the vector z_1 which is available only in step 5). Hence (1.4b) is actually performed after step 5). This means that we may have to repeat steps 4) and 5) after we computed a more accurate solution in step 3), if (1.4b) failed to be satisfied initially.

In practice, it can be advisable to test instead on the sign of the leading coefficient in the conjugate gradient method (which must be positive if the preconditioned operator has a positive definite symmetric part, see [1]). If the sign test is violated, we repeat algorithm 1 with more inner iterations, which corresponds to choosing smaller values of $\varepsilon_1, \varepsilon_2$.

To estimate the deviation of $AB[v]$ from v we note first that Algorithm 1 shows that

$$\begin{aligned} AB[v] - v &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} A_{11}(w_1 - z_1) + A_{12}x_2 \\ A_{21}(w_1 - z_1) + A_{22}x_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} A_{11}w_1 - v_1 - (A_{11}z_1 - y_1) \\ A_{22}x_2 - w_2 - A_{21}z_1 \end{bmatrix} \end{aligned} \tag{1.5}$$

Let

$$\sigma_1 = \|A_{12}S^{-1}\|_0, \quad \sigma_2 = \|A_{21}A_{11}^{-1}\|_0. \tag{1.6}$$

We assume that $\varepsilon_1\sigma_2 < 1$. Note now that (1.4a) shows that

$$\begin{aligned} \|w_2\|_0 &\le \|v_2\|_0 + \|A_{21}A_{11}^{-1}A_{11}w_1\|_0 \le \|v_2\|_0 + \sigma_2\|A_{11}w_1\|_0 \\ &\le \|v_2\|_0 + \sigma_2(1+\varepsilon_1)\|v_1\|_0. \end{aligned} \tag{1.7}$$

Further

$$\begin{aligned} \|y_1\|_0 = \|A_{12}x_2\|_0 &= \|A_{12}S^{-1}SC[w_2]\|_0 \le \sigma_1\|SC[w_2]\|_0 \\ &\le \sigma_1[\|w_2\|_0 + \|SC[w_2] - w_2\|_0]. \end{aligned} \tag{1.8}$$

Now using (1.4 a,b) we get

$$\begin{aligned} &\|SC[w_2] - w_2\|_0 = \|Sx_2 - w_2\|_0 = \\ &\|A_{22}x_2 - A_{21}A_{11}^{-1}A_{12}x_2 - w_2\|_0 = \\ &\|A_{22}x_2 - A_{21}B_{11}[y_1] - w_2 + A_{21}(B_{11}[y_1] - A_{11}^{-1}A_{12}x_2)\|_0 \le \\ &\|A_{22}x_2 - A_{21}z_1 - w_2 + A_{21}A_{11}^{-1}[A_{11}z_1 - y_1]\|_0 \le \\ &\varepsilon_2\|w_2\|_0 + \sigma_2\varepsilon_1\|y_1\|_0. \end{aligned}$$

This, and (1.8) show that

$$\|y_1\|_0 \le \sigma_1[\|w_2\|_0 + \varepsilon_2\|w_2\|_0]/(1-\varepsilon_1\sigma_2)$$

so, together with (1.7) we get

$$(1+\varepsilon_1)\|v_1\|_0]/(1-\varepsilon_1\sigma_2). \tag{1.9}$$

(1.5), (1.7), (1.4a,b) and (1.9) show now that

$$\begin{aligned}
\|AB[v]-v\|_0 &\le \{(\|A_{11}w_1-v_1\|_0+\|A_{11}z_1-y_1\|_0)^2+\|A_{22}x_2-w_2-A_{21}z_1\|_0^2\}^{\frac{1}{2}}\\
&\le \|A_{11}w_1-v_1\|_0+\|A_{11}z_1-y_1\|_0+\|A_{22}x_2-w_2-A_{21}z_1\|_0\\
&\le \varepsilon_1\|v_1\|_0+\varepsilon_1(1+\varepsilon_2)\sigma_1[\|v_2\|_0+\sigma_2(1+\varepsilon_1)\|v_1\|_0]/(1-\varepsilon_1\sigma_2)\\
&\quad +\varepsilon_2\|v_2\|_0+\varepsilon_2(1+\varepsilon_1)\sigma_2\|v_1\|_0\\
&= [\varepsilon_1+\varepsilon_1(1+\varepsilon_1)(1+\varepsilon_2)\sigma_1\sigma_2/(1-\varepsilon_1\sigma_2)+\varepsilon_2(1+\varepsilon_1)\sigma_2]\|v_1\|_0\\
&\quad +[\varepsilon_2+\varepsilon_1(1+\varepsilon_2)\sigma_1/(1-\varepsilon_1\sigma_2)]\|v_2\|_0\\
&\le C(\varepsilon_1,\varepsilon_2,\sigma_1,\sigma_2)\|v\|_0,
\end{aligned}$$

where $\|v\|_0=\{\|v_1\|^2+\|v_2\|^2\}^{\frac{1}{2}}$. Here

$$\begin{aligned}
C(\varepsilon_1\varepsilon_2,\sigma_1,\sigma_2) &\le \sqrt{2}\max\{\varepsilon_1+\varepsilon_1(1+\varepsilon_1)(1+\varepsilon_2)\sigma_1\sigma_2/(1-\varepsilon_1\sigma_2)+\varepsilon_2(1+\varepsilon_1)\sigma_2,\\
&\qquad [\varepsilon_2+\varepsilon_1(1+\varepsilon_2)\sigma_1/(1-\varepsilon_1\sigma_2)]\}.
\end{aligned} \tag{1.10}$$

Note that

$$C(\varepsilon_1,\varepsilon_2,\sigma_1,\sigma_2)<\sqrt{2}[\varepsilon_1+\varepsilon_1\sigma_1+\varepsilon_1\sigma_1\sigma_2+\varepsilon_2+\varepsilon_2\sigma_2],\quad \varepsilon_1,\varepsilon_2\to 0.$$

In [5] it was shown that for $\varepsilon_1,\varepsilon_2$ sufficiently small, $AB[\cdot]$ is coercive and bounded, namely

$$(v,AB[v])_0\ge(1-C^{\frac{1}{2}})\|v\|_0^2,\quad \text{for all } v$$

and

$$\|AB[v]\|_0\le(1+C^{\frac{1}{2}})\|v\|_0^2,\quad \text{for all } v.$$

It was also shown that for the residuals $r^k=Ax^k-b$, $k=0,1,\ldots$ computed by a preconditioned generalized conjugate gradient (GCG) method, or even by a preconditioned steepest descent method, with the variable-step preconditioner $B[\cdot]$, the following convergence rate estimate holds:

$$\|r^k\|_0\le\left(1-\left(\frac{\delta_1}{\delta_2}\right)^2\right)^{\frac{k}{2}}\|r^0\|_0,\quad k\ge 0,$$

where $\delta_1=1-C^{\frac{1}{2}}$, $\delta_2=1+C^{\frac{1}{2}}$.

Note that the above method is parameter free, i.e. for a user the method looks like a "black-box".

In Axelsson and Vassilevski [5] several particular applications of the construction of variable-step preconditioners are demonstrated, namely for two-level hierarchical iterative methods for the finite element discretization of non-self adjoint elliptic problems, for mixed finite element discretization of second order elliptic problems and for Stokes equation. Related results for the latter application were studied earlier by Bank, Welfert and Yserentant [9], by Langer and Queck [17], Verfürth [22] and more recently with application to domain decomposition methods with inexact subdomain solvers by Börgers [11], Langer [16] and Y. Vasilevskij [21]. However, in all of these papers the constructed preconditioner is either a fixed matrix or the global (outer) iterative method is a stationary one, i.e. not of a variational type.

In the present paper we simplify our original method in such a way that only one accurate inner iteration step with the mapping $B_{11}[\cdot]$ is required per outer iteration. This is made possible by use of a parameter free method to construct approximate inverses for the action of an approximate Schur complement $\widetilde{S}v_2 = A_{22}v_2 - A_{21}B_{11}[A_{12}v_2]$ or for the residuals arising from the approximate Schur complement reduced system for a defect corrected global system. Although we work on the reduced system we can show convergence for the global system and with a rate which depends only on the accuracy of the mapping $B_{11}[\cdot]$ and of the approximate inverse of the Schur complement.

Before presenting the new method we consider a general method to construct approximate inverses to nonlinear mappings that are almost linear. This method will then be applied when we compute actions of approximations of the inverses of the Schur complement matrix.

2. Construction of approximate inverses to nonlinear mappings that are almost linear. Consider first the following nonlinear equation,

$$\widetilde{A}[x] = v, \tag{2.1}$$

where the mapping $\widetilde{A}[\cdot]$ is *assumed to be sufficiently close to a linear* in the sense that for some matrix A, we have

$$\|Ax - \widetilde{A}[x]\|_1 \le \delta \|Ax\|_1, \quad \text{for all } x, \tag{2.2}$$

where $\delta \in (0,1)$ is sufficiently small and where $\|x\|_1 = \sqrt{(x,x)_1}$ is a given norm, defined by an inner product $(\cdot,\cdot)_1$. The mappings A and $\widetilde{A}$ can be preconditioned forms of some operators $\widehat{A}$ and $\widehat{\widetilde{A}}$, say, by some preconditioning matrix D. For notational simplicity we do not give this preconditioner in explicit form. As an application of this, in the following section, A will be a Schur complement matrix and $\widetilde{A}$ an approximation of A whose closeness to A is fully controlled by the number of inner iterations used to solve systems with the top matrix block of the original global system.

We assume that A satisfies the following boundedness and coercivity properties, i.e. for some positive constants γ_1, γ_2 we have

$$\|Ax\|_1 \le \gamma_2 \|x\|_1, \quad \text{for all } x, \tag{2.3}$$

$$(Ax, x)_1 \ge \gamma_1 (x,x)_1, \quad \text{for all } x. \tag{2.4}$$

Note that for this to hold the matrix A need not be symmetric.

We consider now the following variational type algorithm to approximately solve the nonlinear equation (2.1).

ALGORITHM 2.

initiate: choose x^0; $r^0 = v - \widetilde{A}[x^0]$;

for $k = 1, 2, \ldots$ compute

$$\begin{aligned}
\widetilde{r}^{k-1} &= \widetilde{A}[r^{k-1}];\\
\alpha_{k-1} &= (r^{k-1}, \widetilde{r}^{k-1})_1/(\widetilde{r}^{k-1}, \widetilde{r}^{k-1})_1;\\
x^k &= x^{k-1} + \alpha_{k-1} r^{k-1};\\
r^k &= v - \widetilde{A}[x^k];
\end{aligned}$$

It can be seen that for a linear mapping, i.e. a matrix $\widetilde{A}$, the above algorithm reduces to the steepest descent method to compute iterations x^k for which we have monotone convergence with steepest descent method decay in the gradient of the functional $(r^k, r^k)_1$.

The next theorem shows that algorithm 2 gives an approximate solution of (2.1), which converges with a geometric rate of convergence.

THEOREM 2.1. *The k-th iterate x^k, $k \geq 1$, generated by Algorithm 2 satisfies the inequality,*

$$\|v - \widetilde{A}[x^k]\|_1 \leq \left[q^k + \frac{1}{1-q}\frac{2\delta}{1-\delta}\right]\|v\|_1,$$

where

$$q = \sqrt{1 - (\widetilde{\gamma}_1/\widetilde{\gamma}_2)^2} + 4\delta/(1-\delta),$$

$\widetilde{\gamma}_1 = \gamma_1 - \delta\gamma_2$, $\widetilde{\gamma}_2 = (1+\delta)\gamma_2$, δ is the constant in (2.2) and where γ_1, γ_2 are defined in (2.3), (2.4), respectively. We assume that δ, $0 < \delta < 1$, is sufficiently small to make $q < 1$, and we see that we can get an arbitrarily accurate solution x^k satisfying

$$\|v - \widetilde{A}[x^k]\|_1 \leq [\varepsilon + O(\delta)]\|v\|_1$$

by choosing δ sufficiently small and performing $k = O(\log \frac{1}{\varepsilon})$ iterations, where ε is the relative stopping accuracy.

Proof. First we show that $\widetilde{A}[\cdot]$ itself is bounded and coercive. Using (2.2), (2.3) and (2.4) one readily derives

$$(\widetilde{A}[x], x)_1 = (\widetilde{A}[x] - Ax, x)_1 + (Ax, x)_1 \geq (\gamma_1 - \delta\gamma_2)(x, x)_1,$$

and

$$\|\widetilde{A}[x]\|_1 \leq (1+\delta)\|Ax\|_1 \leq (1+\delta)\gamma_2\|x\|_1.$$

Hence, for the $(k-1)$st residual r^{k-1} in Algorithm 2, we have

$$(r^{k-1}, \widetilde{A}[r^{k-1}])_1 \geq \widetilde{\gamma}_1\|r^{k-1}\|_1^2 \tag{2.5}$$

$$(\widetilde{A}[r^{k-1}], \widetilde{A}[r^{k-1}])_1 \leq \widetilde{\gamma}_2^2\|r^{k-1}\|_1^2,$$