ROLAND GLOWINSKI,
JACQUES-LOUIS LIONS
AND JIWEN HE

Exact and Approximate Controllability for Distributed Parameter Systems

A Numerical Approach

ENCYCLOPEDIA OF MATHEMATICS

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ROLAND GLOWINSKI
University of Houston

JACQUES-LOUIS LIONS
College de France, Paris

JIWEN HE
University of Houston

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Exact and Approximate Controllability for Distributed Parameter Systems

The behavior of systems occurring in real life is often modeled by partial differential equations. This book investigates how a user or observer can influence the behavior of such systems mathematically and computationally. A thorough mathematical analysis of controllability problems is combined with a detailed investigation of methods used to solve them numerically, these methods being validated by the results of numerical experiments. In Part I of the book, the authors discuss the mathematics and numerics relating to the controllability of systems modeled by linear and nonlinear diffusion equations; Part II is dedicated to the controllability of vibrating systems, typical ones being those modeled by linear wave equations; finally, Part III covers flow control for systems governed by the Navier–Stokes equations modeling incompressible viscous flow. The book is accessible to graduate students in applied and computational mathematics, engineering, and physics; it will also be of use to more advanced practitioners.

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To Andrée, Angela, and April, and to Dorian Lions

LENS LARQUE-homonyms, with definitions.

1. Lencilorqua: a village of 657 inhabitants on Vasselona Continent, Reis, sixth planet to Gamma Eridani.
2. Lanslarke: a predacious winged creature of Dar Sai, third planet of Cora, Argo Navis 961.
3. Laenzle arc: the locus of a point generated by the *seventh theorem of triskoïd dynamics*, as defined by the mathematician Palo Laenzle (907–1070).
4. Linslurk: a mosslike . . .

Jack Vance, *The Face*. In *The Demon Princes*, Volume II,
Tom Doherty Associates, Inc., New York, NY, 1997

The most challenging course I took in high school was calculus.

Bill Clinton, *My Life*, Knopf, New York, NY, 2004

The real trick to writing a book is writing. Until you have a book.

Adam Felber, *Schrödinger's Ball*, Random House, New York, NY, 2006

Preface

During ICIAM 1995, in Hamburg, David Tranah approached Jacques-Louis Lions and myself and asked us if we were interested in publishing in book form our two-part article "*Exact and approximate controllability for distributed parameter systems*" which had appeared in *Acta Numerica 1994* and *1995*. The length of the article (almost 300 pages) was a justification, among several others, for such an initiative. While I was very enthusiastic about this project, J.L. Lions was more cautious, without being against it. Actually, his reservation concerning this book project was stemming from recent important developments on controllability related issues, justifying, in his opinion an in-depth revision of our article. Both of us being quite busy, the project was practically forgotten. As everyone knows in the Scientific Community, and elsewhere, Jacques-Lions passed away in June 2001, while still active scientifically. He largely contributed in making the *Control of Distributed Parameter Systems* a most important field where sophisticated mathematical and computational techniques meet with advanced applications. Therefore, when David Tranah renewed his 1995 suggestion during a conference of the European Mathematical Society held in Nice in February 2003, we thought that it would be a very nice way to pay to J.L. Lions the tribute he fully deserves. The idea was to respect as much as possible the original text, since it largely reflects J.L. Lions' inspired scientific vision, and also its inimitable way at making simple complicated notions. On the other hand, it was also agreed that additional material should be included to make the text more up to date. Most of these additions are concerned with *flow control*; indeed, for J.L. Lions, the control of flow modeled by the Navier–Stokes equations was a kind of scientific Holy Grail and we are most happy that he could witness the first real mathematical and computational successes in that direction, all taking place in the late 1990s.

The present volume is structured as follows:

- Motivations and some broad generalities are given in the Introduction.
- Part I is dedicated to the control of *linear* and *nonlinear diffusion models*; it contains Sections 1–5 of the *Acta Numerica* article, with additional materials such as the Neumann control of unstable advection–reaction–diffusion models, and a discussion of computer memory saving methods for the solution of time-dependent control problems by adjoint-equation-based methods. A short introduction to *Riccati-equation*-based control methods is also provided.

- Part II is concerned with the controllability of *wave equation* type models and of *coupled systems*. This material corresponds essentially to Sections 6 and 7 of the *Acta Numerica* article.
- Part III is the main addition to the original text; it is dedicated to the *boundary* control, by either rotation or blowing and suction, of *Newtonian incompressible viscous flow* modeled by the *Navier–Stokes equations*.

Since most of the additional material follows from investigations conducted jointly with Professor Jiwen He, a former collaborator of J.L. Lions, all the parties involved found it quite natural to have him as a coauthor of this volume.

Acknowledgments and warmest thanks should go first to David Tranah, Ken Blake, and Cambridge University Press for encouraging the publication of this augmented version of the *Acta Numerica* article, and also to Mrs Andrée Lions and Professor Pierre-Louis Lions for their acceptation of this project. The invaluable help of Dr H.L. Juárez (UAM-Mexico City) and of his collaborators (Bety Arce, in particular) is also acknowledged; they converted large parts of a text initially written in Word© to a LaTeX© file, a nontrivial task indeed considering the size of this volume.

Special thanks are due to S. Barck-Holst, M. Berggren, H.Q. Chen, J.M. Coron, J.I. Diaz, S. Gomez, M. Gorman, A.J. Kearsley, B. Mantel, R. Metcalfe, J. Périaux, T.-W. Pan, O. Pironneau, J.-P. Puel, A.M. Ramos, T. Rossi, D. Sorensen, J. Toivanen, and E. Zuazua for very helpful comments and suggestions concerning the additions to the original article (further acknowledgments may be found at the end of this volume; they concern the original *Acta Numerica* article).

We will conclude this preface with further thanks to Cambridge University Press for authorizing the reprinting of the above *Acta Numerica* article in Volume III of J.L. Lions, *Oeuvres Choisies*, SMAI / EDP Sciences, Paris, 2003, a three-volume testimony of the outstanding scientific contributions of Jacques-Louis Lions.

Guanajuato, Mexico Roland Glowinski

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Introduction

I.1 What it is all about?

We consider a system whose *state* is given by the solution y to a partial differential equation (PDE) of evolution, and which contains *control functions*, denoted by v.

Let us write all that in a formal fashion for the time being. The *state equation* is written as

$$\frac{\partial y}{\partial t} + \mathcal{A}(y) = \mathcal{B}v, \tag{I.1}$$

where y is a scalar- or vector-valued function.

In (I.1), $\mathcal{A}$ is a set of partial differential operators (PDOs), linear or nonlinear (at least for the time being). In (I.1), v denotes the *control* and $\mathcal{B}$ maps the "space of controls" into the "state space". It goes without saying that all this has to be made more precise. This will be the task of the following sections.

The PDE (I.1) should include *boundary conditions*. We do not make them explicit here. They are supposed to be contained in the abstract formulation (I.1), where v can be either applied *inside* the domain $\Omega \subset \mathbb{R}^d$, where (I.1) is considered (v is then a *distributed* control), or on the boundary Γ of Ω – or on a part of it (v is then a *boundary* control). If v is applied at points of Ω, v is said to be a *pointwise* control.

One has to add also *initial conditions* to (I.1): if we assume that $t = 0$ is the initial time, then these initial conditions are given by

$$y|_{t=0} = y_0, \tag{I.2}$$

with y_0 being a given element of the state space.

It will be assumed that, given v (in a suitable space), problem (I.1)–(I.2) (and the boundary conditions included in the formulation (I.1)) *uniquely defines a solution*. This solution is a function (scalar- or vector-valued) of $x \in \Omega$, $t > 0$, and of y_0 and v. We shall denote this solution by $y(v)$ $(= \{x, t\} \to y(x, t; v))$. Similarly, we shall denote by $y(t; v)$ the function $x \to y(x, t; v)$. Then, the initial condition (I.2) can be written as

$$y(0; v) = y_0. \tag{I.2*}$$

Remark I.1 The notions to be introduced below can be generalized to situations where the *uniqueness* of the solution to problem (I.1)–(I.2) is *not known*. We are thinking here of the Navier–Stokes equations (and related models) when the flow region Ω is a subset of $\mathbb{R}^3$ (and the *Reynolds number* is sufficiently large).

We can now introduce the notion of *controllability*, either *exact* or *approximate*.

Let $T > 0$ be *given* and let y_T (the *target* function) be a *given element* of the state space. We want to "drive the system" from y_0 at $t = 0$ to y_T at $t = T$, that is, we want to find v such that

$$y(T; v) = y_T. \tag{I.3}$$

If this is possible for *any* target function y_T in the state space, one can say that the system is *controllable* (or *exactly controllable)*. If – as we shall see in most of the examples – condition (I.3) is too strict, it is natural to replace it by the less demanding one

$$y(T; v) \text{ belongs to a "small" neighborhood of } y_T. \tag{I.4}$$

If this is possible, one says that the system is *approximately controllable*; otherwise, the system is *not controllable*.

Before giving precise examples, we want to say a few words concerning the motivation for studying these controllability problems.

I.2 Motivation

There are several aspects that make controllability problems important in practice.

Aspect # 1 At a *given time-horizon*, we want the system under study to behave *exactly* as we wish (or in a manner arbitrary close to it).

Problems of this type are common in Science and Engineering: we would like, for example, to have the temperature (or pressure) of a system equal, or very close, to a given value – globally or locally – at a given time. *Chemical Engineering* is an important source of such problems, a typical example in that direction being the design of *car catalytic converters*; in this example chemical reactions have to take place leading to the "destruction" at a given time-horizon (very small in practice) of the polluting chemicals contained in the exhaust gases (the modeling and numerical simulation of catalytic converter systems are discussed in, for example, Engquist, Gustafsson, and Vreeburg (1978), Friedman (1988, Chapter 7), and Friend (1993)).

Aspect # 2 For *linear* systems, it is known (cf. Russel (1978)) that exact controllability is equivalent to the possibility of *stabilizing* the system.

Stabilization problems abound, in particular in (large) composite structures – the so-called "multibody" systems made of many different parts which can be considered as three-, two-, or one-dimensional and which are linked together by *junctions* and *joints*. The modeling and analysis of such systems are the subject of many interesting studies. We want to mention here the contributions of P.G. Ciarlet and his collaborators (see, for example, Ciarlet, Le Dret, and Nzengwa, 1989, Ciarlet, 1990a,b), and those of Sanchez-Hubert and Sanchez-Palencia (1989), Lagnese, Leugering, and Schmidt (1992, 1994), J. Simo and his collaborators (see, for example, Laursen and Simo, 1993), Park and his collaborators (see, for example, Park, Chiou, and Downer, 1990 and Downer, Park, and Chiou, 1992).

Studying *controllability* is *one* approach to *stabilization* as shown in, for example, J.L. Lions (1988a).

Aspect # 3 (On *controllability* and *reversibility*): Suppose we have a system that *was* in a state z_1 at time $t = -t_0$, $t_0 > 0$, and that is *now* (that is, at $t = 0$) in the state y_0.

We would like to have the system *returning* to a state as close as possible to z_1, that is, $y_T = z_1$. If this is possible, it means some kind of "reversibility" property for the system under consideration. What we have in mind here are *environmental systems*; should they be "local" or "global" in the space variables?

Noncontrollable (sub)systems can suffer "irreversible" changes (cf. J.L. Lions, 1990 and Diaz, 1991).

We return now to the general questions of Section I.1, making them more precise before giving examples.

I.3 Topologies and numerical methods

The topology of the state space appears explicitly in condition (I.4). It is obvious that approximate controllability *depends* on the choice of the topology on the state space, that is, of the state space itself. Actually, *exact* controllability depends on the choice of the state space as well. The choice of the state space is therefore an obviously fundamental issue for the *theory*. We want to emphasize that it is also a fundamental issue from the *numerical point of view*. Indeed, if one has (as we shall see in several situations) exact or approximate controllability in a very general space (which can include elements that are not distributions but "ultra-distributions") but *not* in a classical space of smooth (or sufficiently smooth) functions, then the numerical approximation will *necessarily* develop singularities; "remedies" should be based on the knowledge of the topology where the theoretical convergence is taking place. We shall return on these issues in the following sections; actually, some of them have been addressed in, for example, Dean, Glowinski, and Li (1989), Glowinski and Li (1990), Glowinski, Li, and Lions (1990), Glowinski (1992a), where various *filtering* techniques are discussed in order to eliminate the numerical singularities mentioned above.

In the next section we shall address the following question (of general nature also), namely,

How to choose the control?

I.4 Choice of the control

Let us return to the general formulation (I.1), (I.2), (I.3), (or (I.4)). If there exists, *one* control v achieving these conditions, then there exist, in general, *infinitely many other* controls, vs, also achieving these conditions. Which one should we choose and how?

A most important question is: how to *norm* (we are always working in Banach or Hilbert spaces) the vs? This is related to the *topology* of the state space. It is indeed clear that the regularity (or irregularity!) properties of v and y in (I.1) are related. Let us assume that a norm $v \mapsto |||v|||$ is chosen. Once this choice is made, a natural formulation of the problem is then to find

$$\inf |||v|||, \tag{I.5}$$

among all those vs such that (I.1), (I.2), (I.3), or (I.4) take place.

Remark I.2 There is still some flexibility here since problem (0.5) still makes sense if one replaces $|||\cdot|||$ by a *stronger* norm. This remark will be of practical interest as we shall see later on.

Remark I.3 One can encounter questions of controllability for systems depending on "small" parameters. Two classical (by now) examples are

(i) *Singular perturbations*.
(ii) *Homogenization* which is important for the controllability of structures made of *composite materials*.

In these situations one has to introduce either *families* of norms in (I.5) or norms *equivalent* to $|||\cdot|||$, but which depend on the homogenization parameter.

I.5 Relaxation of the controllability notion

Let us return again to (I.1) and (I.2):

Condition (I.3) concerns the state y itself. In a "complex system" this condition can be (and will be in general) unnecessarily strong. We may want some *subsystems* to behave according to our wishes. We may also want some *average* values to behave accordingly, and so on. A general formulation is as follows:

We consider an operator

$$C \in \mathcal{L}(Y, \mathcal{H}), \tag{I.6}$$

where Y is the state space (chosen!) and where $\mathcal{H}$ is another Banach or Hilbert space (the *observation* space). Think, for instance, of C as being an *averaging* operator. Then, we "relax" condition (I.3) (respectively (I.4)) as follows:

$$Cy(T; v) = h_T, \quad h_T \text{ given in } \mathcal{H} \tag{I.7}$$