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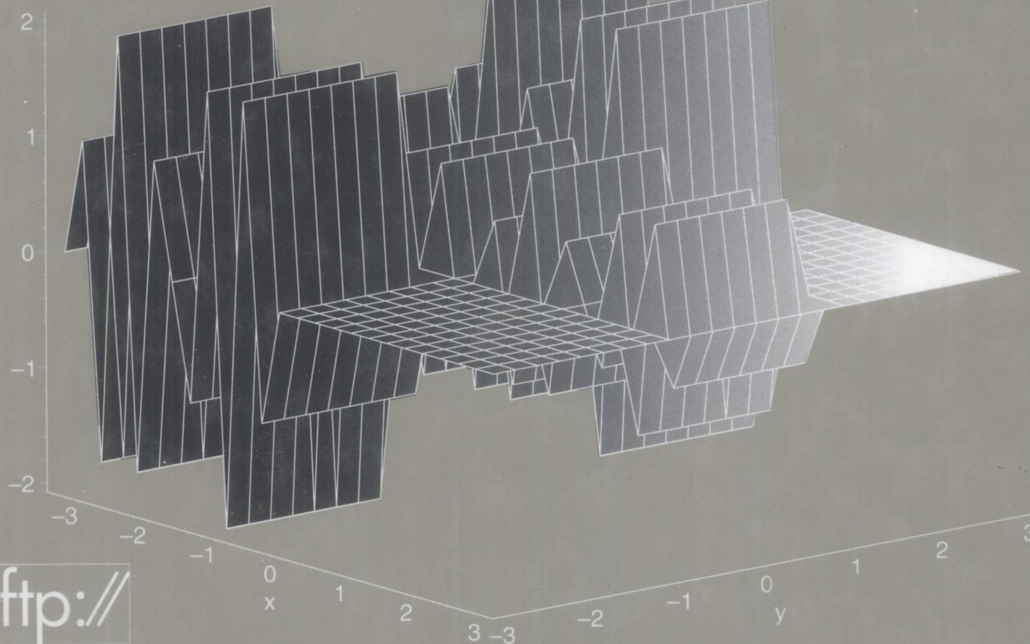
Partial Differential Equations of Applied Mathematics

Erich Zauderer

Third Edition



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PARTIAL DIFFERENTIAL EQUATIONS OF APPLIED MATHEMATICS

Third Edition

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To my wife, Naomi,
my children,
and my grandchildren

PREFACE

The study of partial differential equations (PDEs) of applied mathematics involves the formulation of problems that lead to partial differential equations, the classification and characterization of equations and problems of different types, and the examination of exact, approximate, and numerical methods for the solution of these problems. Each of these aspects is considered in this book.

The widespread availability of computers in the scientific community and the advent of mathematical software such as *Maple, Matlab,* and *Mathematica* has had the effect of eliminating the need for carrying out many routine symbolic and numerical calculations that arise when solving PDEs manually. Furthermore, it has become possible to create and employ fairly sophisticated numerical methods for the solution of PDEs without having to use lengthy computer codes created by professional numerical analysts. For example, *Maple* has built-in procedures or codes that can solve both ordinary differential equations (ODEs) and PDEs symbolically and numerically. The procedures available for the solution of initial and boundary value problems for ODEs greatly exceed those that are available for the solution of initial and boundary value problems for PDEs. For that reason we have created a number of *Maple* procedures that deal with problems arising in the solution of PDEs and are related to the material in each of the chapters in the book. These procedures generate solutions to problems using the methods developed in each chapter. A graphical representation of the results can often be generated. This has been done for the first ten chapters of the book, whose material generally follows the presentation of the second edition of the text.

Two new chapters dealing with *finite difference* and *finite element methods* have been added for the third edition. For these two chapters a large number of *Maple procedures* have been created for the solution of various initial and boundary value problems for PDEs. Thus, not only are the ideas behind the numerical solution methods presented, but their implementation is made possible. It was not possible to do everything in triplicate, using *Maple*, *Matlab*, and *Mathematica*, because many new codes were created. As a result, it was decided to restrict our presentation to the use of *Maple*.

The first chapter is concerned with the formulation of problems that give rise to first- and second-order PDEs representative of the three basic types (parabolic, hyperbolic, and elliptic) considered in this book. These equations are all obtained as limits of difference equations that serve as models for discrete *random walk problems*. These problems are of interest in the theory of Brownian motion and this relationship is examined. A new section has been added that presents random walks that yield first order PDEs in the limit. Finally, a section that employs *Maple* procedures to simulate the various random walks and thereby generate approximate solutions of the related PDEs is included. These methods fall under the general heading of *Monte Carlo methods*. They represent an alternative to the direct numerical solution of the difference equations in the manner considered in Chapter 11. Only elementary concepts from probability theory are used in this chapter.

Chapter 2 deals with first order PDEs and presents the *method of characteristics* for the solution of initial value problems for these equations. Problems that arise or can be interpreted in a wave propagation context are emphasized. First order equations also play an important role in the methods presented in Chapters 9 and 10.

In Chapter 3, PDEs are classified into different types and simplified *canonical forms* are obtained for second order linear equations and certain first order systems in two independent variables. The concept of *characteristics* is introduced for higher-order equations and systems of equations, and its significance for equations of different types is examined. In addition, the question of what types of *auxiliary conditions* are to be placed on solutions of PDEs so that the resulting problems are reasonably formulated is considered. Further, some physical concepts, such as *energy conservation* and *dispersion*, which serve to distinguish equations of different types are discussed. Finally, the concept of *adjoint differential operators* is presented.

Chapter 4 presents the method of *separation of variables* for the solution of problems given in bounded spatial regions. This leads to a discussion of *eigenvalue problems* for PDEs and the one-dimensional version thereof, known as the *Sturm-Liouville problem*. *Eigenfunction expansions*, in general, and *Fourier series*, in particular, are considered and applied to the solution of homogeneous and inhomogeneous problems for linear PDEs of second order. It is also shown that eigenfunction expansions can be used for the solution of *nonlinear problems* by considering a nonlinear heat conduction problem.

In Chapter 5, the *Fourier, Fourier sine, Fourier cosine, Hankel*, and *Laplace transforms* are introduced and used to solve various problems for PDEs given over unbounded regions in space or time. As the solutions of these problems are generally obtained in an integral form that is not easy to evaluate, *approximation methods* for the evaluation of Fourier and Laplace integrals are presented.

Not all problems encountered in applied mathematics lead to equations with smooth coefficients or have solutions that have as many derivatives as required by the order of the PDEs. Consequently, Chapter 6 discusses methods whereby the concept of solution is weakened by replacing the PDEs by *integral relations* that reduce the number of derivatives required of solutions. Also, methods are presented for dealing with problems given over *composite media* that can result in singular coefficients. Finally, the method of *energy integrals* is discussed and shown to yield information regarding the uniqueness and dependence on the data of solutions of PDEs.

Green's functions, which are discussed in Chapter 7, depend on the theory of generalized functions for their definition and construction. Therefore, a brief but self-contained discussion of *generalized functions* is presented in this chapter. Various methods for determining Green's functions are considered and it is shown how initial and boundary value problems for PDEs can be solved in terms of these functions.

Chapter 8 contains a number of topics. It begins with a *variational characterization* of the eigenvalue problems considered in Chapter 4, and this is used to verify and prove some of the properties of eigenvalues and eigenfunctions stated in Chapter 4. Furthermore, the *Rayleigh-Ritz method*, which is based on the variational approach, is presented. It yields an approximate determination of eigenvalues and eigenfunctions in cases where exact results are unavailable. The classical *Riemann method* for solving initial value problems for second order hyperbolic equations is discussed briefly, as are *maximum* and *minimum principles* for equations of elliptic and parabolic types. Finally, a number of basic *PDEs of mathematical physics* are studied, among which the equations of fluid dynamics and Maxwell's equations of electromagnetic theory are discussed at length.

Chapters 9 and 10 deal with *perturbation* and *asymptotic methods* for solving both linear and nonlinear PDEs. In recent years these methods have become an important tool for the applied mathematician in simplifying and solving complicated problems for linear and nonlinear equations. *Regular* and *singular perturbation methods* and *boundary layer theory* are discussed in Chapter 9. Linear and nonlinear *wave propagation problems* associated with the reduced wave equation that contains a large parameter are examined in Chapter 10. These include the scattering and diffraction of waves from various obstacles and the problem of beam propagation in *linear* and *nonlinear optics*. It is also shown in Chapter 10 how singularities that can arise for solutions of hyperbolic equations can be analyzed without having to solve the full problem given for these equations. Finally, an *asymptotic simplification* procedure is presented that permits the replacement of linear and nonlinear equations and systems by simpler equations that retain certain essential features of the solutions of the original equations.

Chapter 11 presents a full discussion of *finite difference methods* for the numerical solution of initial and initial and boundary value problems for PDEs. Equations of all three types, as well as systems of PDEs, are considered. Linear and nonlinear problems are examined. A large number of difference schemes are introduced, and questions of consistency and stability are examined. Specially created and built-in *Maple procedures* are presented for implementation of most of these difference schemes.

The *finite element method* for the approximate numerical solution of initial and initial and boundary value problems for a large class of PDEs in two spatial dimensions is presented in Chapter 12. It is developed from the *Galerkin integral representations* of the given problems and the *Galerkin method* for constructing approximate solutions of these problems. *Triangulations* of the spatial region over which the problem is formulated are created and *finite element solutions* are constructed. *Maple procedures* that carry out these processes are presented and their use is demonstrated.

The text includes a substantial number of figures. As we have indicated, not only do built-in and the newly constructed *Maple* procedures solve problems analytically and numerically, but they can also represent results graphically. The figures in Chapters 1, 11, and 12, were generated with the use of *Maple*, which was not the case for the remaining figures. This accounts for the difference in the representation of coordinate axes in some of the figures, for example.

The Bibliography contains a list of references as well as additional reading. The entries are arranged according to the chapters of the book and they provide a collection of texts and papers that discuss some or all of the material covered in each chapter, possibly at a more elementary or advanced level than that of the text.

This book is intended for advanced undergraduate and beginning graduate students in applied mathematics, the sciences, and engineering. The student is assumed to have completed a standard calculus sequence including elementary ODEs, and to be familiar with some elementary concepts from advanced calculus, vector analysis, and matrix theory. (For instance, the concept of uniform convergence, the divergence theorem, and the determination of eigenvalues and eigenvectors of a matrix are assumed to be familiar to the student.) Although a number of equations and problems considered are physically motivated, a knowledge of the physics involved is not essential for the understanding of the mathematical aspects of the solution of these problems.

In writing this book I have not assumed that the student has been previously exposed to the theory of PDEs at some elementary level and that this book represents the next step. Thus I have included such standard solution techniques as the separation of variables and eigenfunction expansions together with the more advanced methods described earlier. However, in contrast to the more elementary presentations of this subject, this book does not dwell at great length on the method of separation of variables, the theory of Fourier series or integrals, the Laplace transform, or the theory of Bessel or Legendre functions. Rather, the standard results and methods are presented briefly but from a more general and advanced point of view. Thus, even with the addition of the numerical finite difference and finite element methods, it has been possible to present a variety of approaches and methods for solving problems for linear and nonlinear equations and systems without having the length of the book become excessive.

There is more than enough material in the book to be covered in a year-long course. For a shorter course it is possible to use the first part of Chapter 3 and Chapters 4 and 5 as a core, and to select additional material from the other chapters, such as numerical methods, as time permits. The book contains many examples. Very often, new approaches or methods are brought out in the form of an example. Thus the examples should be accorded the same attention as the remainder of the text.

In preparing the third edition, the material contained in the second edition was retained, but rewritten, clarified, and revised where necessary, with corrections made as needed. In addition to the inclusion of two new chapters, some new material was added throughout the first ten chapters. In particular, *Maple* methods that deal with the material in each chapter are presented in a new section at the end of each chapter. Additionally, for example, Chapter 1 has a new discussion of random walks related to first order PDEs. To assist the reader, the sections of the book have been broken up into a collection of subsections that focus on specific topics and subtopics that are considered.

A number of new exercises have been created to supplement those of the second edition. The exercises are placed at the end of each section. With a few exceptions, no substantially new theories or concepts are introduced in the exercises. For the most part, the exercises are based on material developed in the text, and the student should attempt to solve as many of them as possible to test his or her mastery of the subject. Answers and solutions to selected exercises and all the Maple codes that were created for use in the book are available via the FTP site:

ftp://ftp.wiley.com/sci_tech_med/partial_differential/

A supplementary Instructor's Solutions Manual is also available.

I would like to thank Susanne Steitz and Steve Quigley, mathematics editors at Wiley-Interscience, for their support of this project. I acknowledge my gratitude to my wife, Naomi, for her assistance and understanding during the many hours that were spent in writing this book.

ERICH ZAUDERER

New Jersey
March 2006

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