Frank Beichelt

Stochastic Processes in Science, Engineering and Finance

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Stochastic Processes in Science, Engineering and Finance

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zum 90. Geburtstag

PREFACE

The book is a self-contained introduction into stochastic processes with special emphasis on their applications in science, engineering, finance, computer science and operations research. It provides theoretical foundations for modeling time-dependent random phenomena in these areas and illustrates their application through the analysis of numerous, practically relevant examples. As a nonmeasure theoretic text, the material is presented in a comprehensible, application-oriented way. Its study only assumes a mathematical maturity which students of applied sciences acquire during their undergraduate studies in mathematics and probability theory. However, readers with this background are not advised to completely ignore the introductory chapter 1. Although this chapter mainly summarizes standard knowledge on random variables, it also deals with subjects as mixed probability distributions, moment generating functions, nonparametric classes of probability distributions, inequalities, convergence criteria, and limit theorems, which undergraduate courses usually not cover. For readers who need to refresh their knowledge of probability theory studying chapter 1 is imperative anyway. The study of stochastic processes as of any other mathematically based science requires less routine effort, but more creative work on one's own. Therefore, numerous exercises have been added to enable readers to assess to which extent they have grasped the subject. Solutions to most of the exercises can be found in an appendix or exercises are given together with their solutions. To make the book attractive to theoretically interested readers as well, some important proofs and challenging examples and exercises have been included. Exercises marked with a star belong to this category. The chapters are organized in such a way that reading a chapter only requires knowledge of some of the previous ones. The book has been developed as a course text for undergraduates. But parts of it may also serve as a basis for preparing senior undergraduate and graduate level courses for students of applied fields.

Generally, this book does not deal with data analysis aspects of stochastic processes. It can be anticipated that readers will use a software package for tackling statistical and numerical problems. However, studying the text will enable readers to work more creatively with such software and develop their own one. On the whole, the author hopes the book will fulfil its main purpose which is to enable readers to apply stochastic modeling in their own fields.

This book has its origin in the author's text *Stochastische Prozesse für Ingenieure* [6] (English translation [7]). This is still visible, but substantial changes in contents and presentation have led to a new work. Helpful comments are welcome and should be sent to the author: Beichelt@stats.wits.ac.za.

Johannesburg, November 2005 *Frank Beichelt*

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Frank E. Beichelt

Short Biography of Frank E. Beichelt

Frank E. Beichelt is a professor of operations research in the School of Statistics and Actuarial Science at the University of the Witwatersrand Johannesburg, Republic of South Africa. Previously he was a full professor of mathematics and head of the Department of Mathematics at the University of Applied Sciences Mittweida, Germany, an associate professor for reliability and maintenance theory at the University for Transportation and Communication 'Friedrich List' Dresden, and a deputy head engineer for science and technology in the German lignite mining industry. He holds an MSc in mathematics from the Friedrich-Schiller-University Jena, a Dr. rer. nat. in mathematics from the Mining Academy Freiberg, and a senior doctorate in engineering (Dr. sc. techn.) from the University of Transportation and Communication 'Friedrich List' Dresden. He is author and coauthor of 10 books and about 120 publications in journals and conference proceedings in the field of stochastic modeling.

SYMBOLS AND ABBREVIATIONS

□ ■ ●	symbols after an example, a theorem, a definition
$f(t) \equiv c$	$f(t) = c$ for all $t \in \mathbf{T}$
$f * g$	convolution of two functions f and g
$f^{*(n)}$	nth convolution power of f
$\hat{f}(s),\ L\{f\}$	Laplace transform of a function f
$o(x)$	Landau order symbol
δ_{ij}	Kronecker symbol
x_+	$\max(0, x)$

Probability Theory

$X,\ Y,\ Z$	random variables
$E(X),\ Var(X)$	mean (expected) value of X, variance of X
$f_X(x),\ F_X(x)$	probability density function, (cumulative probability) distribution function of X
$F_Y(y\|x),\ f_Y(y\|x)$	conditional distribution function, density of Y given $X = x$
$X_t,\ F_t(x)$	residual lifetime of a system of age t, distribution function of X_t
R_t	residual lifetime of a system which is operating at time t
$E(Y\|x)$	conditional mean value of Y given $X = x$
$\lambda(x),\ \Lambda(x)$	failure rate, integrated failure rate (hazard function)
$N(\mu, \sigma^2)$	normally distributed random variable (normal distribution) with mean value μ and variance σ^2
$\varphi(x),\ \Phi(x)$	probability density function, distribution function of a standard normal random variable
$f_{\mathbf{X}}(x_1, x_2, \ldots, x_n)$	joint probability density function of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$
$F_{\mathbf{X}}(x_1, x_2, \ldots, x_n)$	joint distribution function of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$
$Cov(X, Y),\ \rho(X, Y)$	covariance, correlation between X and Y
$M(z)$	z-transform (moment generating function) of a discrete random variable or its probability distribution

Stochastic Processes

$\{X(t), t \in \mathbf{T}\},\ \{X_t, t \in \mathbf{T}\}$	continuous-time, discrete-time stochastic process with parameter space $\mathbf{T}$
$\mathbf{Z}$	state space of a stochastic process
$f_t(x),\ F_t(x)$	probability density, distribution function of $X(t)$
$f_{t_1, t_2, \ldots, t_n}(x_1, x_2, \ldots, x_n),\ F_{t_1, t_2, \ldots, t_n}(x_1, x_2, \ldots, x_n)$	joint probability density, joint distribution function of $(X(t_1), X(t_2), \ldots, X(t_n))$

$m(t)$	trend function of a stochastic process
$C(s,t)$	covariance function of a stochastic process
$C(\tau)$	covariance function of a stationary stochastic process
$C(t),\ \{C(t),\ t \ge 0\}$	compound random variable, compound stochastic process
$\rho(s,t)$	correlation function of a stochastic process
$\{T_1, T_2, ...\}$	random point process
$\{Y_1, Y_2, ...\}$	sequence of interarrival times, renewal process
N	integer-valued random variable, discrete stopping time
$\{N(t),\ t \ge 0\}$	(random) counting process
$N(s,t)$	increment of a counting process in $(s, t]$
$H(t),\ H_1(t)$	renewal function of an ordinary, delayed renewal processs
$A(t)$	forward recurrence time, point avaliability
$B(t)$	backward recurrence time
$R(t),\ \{R(t),\ t \ge 0\}$	risk reserve, -process; repair cost rate, -process
R_t	residual lifetime of a system operating at time t
$A,\ A(t)$	stationary (long-run) availability, point availability
$p_{ij},\ p_{ij}(t)$	transition probabilities of a homogeneous, discrete-, continuous-time Markov chain
$q_{ij},\ q_i$	conditional, unconditional transition rates (transition intensities) of a homogeneous, continuous-time Markov chain
$\{\pi_i;\ i \in \mathbf{Z}\}$	stationary state distribution of a homogeneous Markov chain
$\lambda_j,\ \mu_j$	birth, death rates
$\lambda,\ \mu,\ \rho$	arrival rate, service rate, traffic intensity λ/μ (in queueing models)
μ_i	mean sojourn time of a semi-Markov process in state i
μ	drift parameter of a Brownian motion process with drift
W	waiting time in a queueing system
L	lifetime, cycle length, queue length, continuous stopping time
$L(x)$	first-passage time with regard to level x
$L(a,b)$	first-passage time with regard to level $\min(a, b)$
$\{B(t),\ t \ge 0\}$	Brownian motion (process)
σ^2	$\sigma^2 = Var(B(1))$ (volatility)
$\{S(t),\ t \ge 0\}$	standardized Brownian motion ($\sigma = 1$)
$\{\bar{B}(t),\ 0 \le t \le 1\}$	Brownian bridge
$\{D(t),\ t \ge 0\}$	Brownian motion with drift
$\{D_u(t),\ t \ge 0\}$	shifted Brownian motion with drift
$M(t)$	absolute maximum of the Brownian motion (with drift) in $[0, t]$
M	absolute maximum of the Brownian motion (with drift) in $[0, \infty)$
$\{U(t),\ t \ge 0\}$	Ornstein-Uhlenbeck process, integrated Brownian motion process

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CHAPTER 1

Probability Theory

1.1 RANDOM EVENTS AND THEIR PROBABILITIES

Probability theory comprises mathematically based theories and methods for investigating random phenomena. Formally, random phenomena occur in connection with random experiments. A *random experiment* is characterized by two properties:

1. Repetitions of the experiment, even if carried out under identical conditions, generally have different outcomes.

2. The possible outcomes of the experiment are known.

Thus, the outcomes of a random experiment cannot be predicted with certainty. However, if random experiments are repeated sufficiently frequently under identical conditions, *stochastic* or *statistical regularities* can be found. Examples of random experiments are:

1) Counting the number of vehicles arriving at a filling station a day.

2) Counting the number of shooting stars during a fixed time interval. The possible outcomes are, as in the previous random experiment, nonnegative integers.

3) Recording the daily maximum wind velocity at a fixed location.

4) Recording the lifespans of technical systems or organisms.

5) Recording the daily maximum fluctuation of share prices. The possible outcomes are, as in the random experiments 3 and 4, nonnegative numbers.

6) The total profit sombody makes with his financial investments a year. This 'profit' can be negative, i.e. any real number can be the outcome.

As the examples show, in this context the term 'experiment' has a more abstract meaning than in the customary sense.

Random Events A possible outcome a of a random experiment is called an *elementary* or a *simple event.* The set of all elementary events is called *space of elementary events* or *sample space.* Here and in what follows, the sample space is denoted as **M**. A sample space **M** is *discrete* if it is a finite or a countably infinite set.

A *random event* (briefly: *event*) A is a subset of **M**. An event A is said to have *occurred* if the outcome a of the random experiment is an element of A: $a \in A$.

Let A and B be two events. Then the set-theoretic operations *intersection* '$\cap$' and *union* '$\cup$' can be interpreted in the following way:

$A \cap B$ is the event that both A and B occur and $A \cup B$ is the event that A or B (or both) occur.

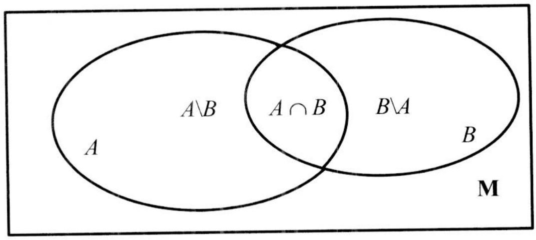

Figure 1.1 Venn Diagram

If $A \subseteq B$ (A is a subset of B), then the occurrence of A implies the occurrence of B.

$A \backslash B$ is the set of all those elementary events which are elements of A, but not of B. Thus, $A \backslash B$ is the event that A occurs, but not B. Note that $A \backslash B = A \backslash (A \cap B)$.

The event $\overline{A} = \mathbf{M} \backslash A$ is the *complement of A*. If A occurs, then $\overline{A}$ cannot occur and vice versa.

Rules of de Morgan Let $A_1, A_2, ..., A_n$ be a sequence of random events. Then

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A}_i, \quad \overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A}_i. \tag{1.1}$$

In particular, if $n = 2$, $A_1 = A$ and $A_2 = B$, the rules of de Morgan simplify to

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}. \tag{1.2}$$

The empty set $\varnothing$ is the *impossible event,* since, for not containing an elementary event, it can never occur. By definition, **M** contains all elementary events so that it must always occur. Hence **M** is called the *certain event.* Two events A and B are called disjoint or (*mutually*) *exclusive* if their joint occurrence is impossible, i.e. if $A \cap B = \varnothing$. In this case the occurrence of A implies that B does not occur and vice versa. In particular, A and $\overline{A}$ are disjoint events (Figure 1.1).

Probability Let $\boldsymbol{M}$ be the set of all those random events A which can occur when carrying out the random experiment, including **M** and $\varnothing$. Further, let $P = P(A)$ be a function on $\boldsymbol{M}$ with properties

I) $P(\varnothing) = 0, \quad P(\mathbf{M}) = 1,$

II) for any event A, $0 \le P(A) \le 1$,

III) for any sequence of disjoint (mutually exclusive) random events $A_1, A_2, ...,$ i.e. $A_i \cap A_j = \varnothing$ for $i \neq j$,

$$P\left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i). \tag{1.3}$$

The number $P(A)$ is the *probability* of event A. $P(A)$ characterizes the degree of certainty of the occurrence of A. This interpretation of the probability is justified by the following implications from properties I) to III).

1) $P(\overline{A}) = 1 - P(A)$.

2) If $A \subseteq B$, then $P(B \setminus A) = P(B) - P(A)$. In this case, $P(A) \le P(B)$.
For any events A and B, $P(B \setminus A) = P(B) - P(A \cap B)$.

3) If A and B are disjoint, i.e. $A \cap B = \varnothing$, then

$$P(A \cup B) = P(A) + P(B).$$

4) For any events A, B, and C,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B), \tag{1.4}$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C)$$
$$+P(A \cap B \cap C).$$

5) In generalizing implications 4), one obtains the *Inclusion-Exclusion-Formula*: For any random events $A_1, A_2, ..., A_n$,

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \Sigma_{k=0}^{n-1} (-1)^{k+1} P_k$$

with

$$P_k = \sum_{i_1 < i_2 < \cdots < i_k}^{n} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}),$$

where the summation runs over all k-dimensional vectors

$$(i_1, i_2, ..., i_k) \text{ with } 1 \le i_1 < i_2 < \cdots < i_k \le n.$$

Note It is assumed that all those subsets of **M** which arise from applying operations $\cap, \cup$ and \ to any random events are also random events, i.e. elements of ***M***.

The probabilities of random events are usually unknown. However, they can be estimated by their relative frequencies. If in a series of n repetitions of one and the same random experiment the event A has been observed $m = m(A)$ times, then the *relative frequency* of A is given by

$$\hat{p}_n(A) = \frac{m(A)}{n}.$$

Generally, the relative frequency of A tends to $P(A)$ as n increases:

$$\lim_{n \to \infty} \hat{p}_n(A) = P(A). \tag{1.5}$$

Thus, the probability of A can be estimated with any required level of accuracy from its relative frequency by sufficiently frequently repeating the random experiment (see section 1.9.2).

Conditional Probability Two random events A and B can depend on each other in the following sense: The occurrence of B will change the probability of the occurrence of A and vice versa. Hence, the additional piece of information 'B has occurred' should be used to predict the occurrence of A more precisely. This is done by defining the conditional probability of A given B.