Advanced Courses in Mathematics CRM Barcelona

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Numerical Solutions of Partial Differential Equations

CENTRE DE RECERCA MATEMÀTICA

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Birkhäuser
Basel · Boston · Berlin

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2000 Mathematical Subject Classification 35-99, 65-99

Library of Congress Control Number: 2008940758

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 978-3-7643-8939-0 Birkhäuser Verlag AG, Basel – Boston – Berlin

Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Printed in Germany

ISBN 978-3-7643-8939-0

e-ISBN 978-3-7643-8940-6

9 8 7 6 5 4 3 2 1

www.birkhauser.ch

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Foreword

This book contains an expanded and smoothed version of lecture notes delivered by the authors at the Advanced School on *Numerical Solutions of Partial Differential Equations: New Trends and Applications*, which took place at the Centre de Recerca Matemàtica (CRM) in Bellaterra (Barcelona) from November 15th to 22nd, 2007.

The book has three parts. The first part, by Silvia Bertoluzza and Silvia Falletta, is devoted to the use of wavelets to derive some new approaches in the numerical solution of PDEs, showing in particular how the possibility of writing equivalent norms for the scale of Besov spaces allows to write down some new methods. The second part, by Giovanni Russo, provides an overview of the modern finite-volume and finite-difference shock-capturing schemes for systems of conservation and balance laws, with emphasis in giving a unified view of such schemes by identifying the essential aspects of their construction. In the last part Chi-Wang Shu gives a general introduction to the discontinuous Galerkin methods for solving some classes of PDEs, discussing cell entropy inequalities, nonlinear stability and error estimates.

The school that originated these notes was born with the objective of providing an opportunity for PhD students, recent PhD doctorates and researchers in general in fields of applied mathematics and engineering to catch up with important developments in the fields and/or to get in touch with state-of-the-art numerical techniques that are not covered in usual courses at graduate level.

We are indebted to the Centre de Recerca Matemàtica and its staff for hosting the advanced school and express our gratitude to José A. Carrillo (Institució Catalana de Recerca i Estudis Avançats – Universitat Autònoma de Barcelona), Rosa Donat (Universitat de Valéncia), Carlos Parés (Universidad de Málaga) and Yolanda Vidal (Universitat Politècnica de Catalunya) for the mathematical organisation of the course and for making it such a pleasant experience.

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Edited by
Manuel Castellet

Since 1995 the Centre de Recerca Matemàtica (CRM) in Barcelona has conducted a number of annual Summer Schools at the post-doctoral or advanced graduate level. Sponsored mainly by the European Community, these Advanced Courses have usually been held at the CRM in Bellaterra.
The books in this series consist essentially of the expanded and embellished material presented by the authors in their lectures.

Argyros, S. / Todorcevic, S.
Ramsey Methods in Analysis (2005)
ISBN 978-3-7643-7264-4

Audin, M. / Cannas da Silva, A. / Lerman, E.
Symplectic Geometry of Integrable Hamiltonian Systems (2003)
ISBN 978-3-7643-2167-3

Bertoluzza, S. / Falletta, S. / Russo, G. / Shu, C.-W.
Numerical Solutions of Partial Differential Equations (2008)
ISBN 978-3-7643-8939-0

Brady, N. / Riley, T. / Short, H.
The Geometry of the Word Problem for Finitely Generated Groups (2006)
ISBN 978-3-7643-7949-0

The origins of the word problem are in group theory, decidability and complexity, but, through the vision of Gromov and the language of filling functions, the topic now impacts the world of large-scale geometry, including topics such as soap films, isoperimetry, coarse invariants and curvature.
The first part introduces van Kampen diagrams in Cayley graphs of finitely generated, infinite groups; it discusses the van Kampen lemma, the isoperimetric functions or Dehn functions, the theory of small cancellation groups and an introduction to hyperbolic groups. The second part is dedicated to Dehn functions, negatively curved groups, in particular, CAT(0) groups, cubings and cubical complexes. In the last part, filling functions are presented from geometric, algebraic and algorithmic points of view. Many examples and open problems are included.

Brown, K.A. / Goodearl, K.R.
Lectures on Algebraic Quantum Groups (2002)
ISBN 978-3-7643-6714-5

Catalano, D. / Cramer, R. / Damgård, I. / Di Creszenso, G. / Pointcheval, D. / Takagi, T.
Contemporary Cryptology (2005)
ISBN 978-3-7643-7294-1

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Global Riemannian Geometry: Curvature and Topology (2003)
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Mislin, G. / Valette, A.
Proper Group Actions and the Baum-Connes Conjecture (2003)
ISBN 978-3-7643-0408-9

Myasnikov, A. / Shpilrain, V. / Ushakov, A.
Group-based Cryptography (2008)
ISBN 978-3-7643-8826-3

This book is about relations between three different areas of mathematics and theoretical computer science: combinatorial group theory, cryptography, and complexity theory. It is explored how non-commutative (infinite) groups, which are typically studied in combinatorial group theory, can be used in public key cryptography. It is also shown that there is a remarkable feedback from cryptography to combinatorial group theory because some of the problems motivated by cryptography appear to be new to group theory, and they open many interesting research avenues within group theory.
Then, complexity theory, notably generic-case complexity of algorithms, is employed for cryptanalysis of various cryptographic protocols based on infinite groups, and the ideas and machinery from the theory of generic-case complexity are used to study asymptotically dominant properties of some infinite groups that have been applied in public key cryptography so far.
Its elementary exposition makes the book accessible to graduate as well as undergraduate students in mathematics or computer science.

Contents

Part I

Wavelets and Partial Differential Equations

Silvia Bertoluzza and Silvia Falletta

Introduction

Wavelet bases were introduced in the late 1980s as a tool for signal and image processing. Among the applications considered at the beginning we recall applications in the analysis of seismic signals, the numerous applications in image processing – image compression, edge-detection, denoising, applications in statistics, as well as in physics. Their effectiveness in many of the mentioned fields is nowadays well established: as an example, wavelets are actually used by the US *Federal Bureau of Investigation* (or FBI) in their fingerprint database, and they are one of the ingredients of the new MPEG media compression standard. Quite soon it became clear that such bases allowed to represent objects (signals, images, turbulent fields) with singularities of complex structure with a low number of degrees of freedom, a property that is particularly promising when thinking of an application to the numerical solution of partial differential equations: many PDEs have in fact solutions which present singularities, and the ability to represent such a solution with as little as possible degrees of freedom is essential in order to be able to implement effective solvers for such problems. The first attempts to use such bases in this framework go back to the late 1980s and early 1990s, when the first simple adaptive wavelet methods [32] appeared. In those years the problems to be faced were basic ones. The computation of integrals of products of derivatives of wavelets – objects which are naturally encountered in the variational approach to the numerical solution of PDEs – was an open problem (solved later by Dahmen and Michelli in [25]). Moreover, wavelets were defined on $\mathbb{R}$ and on $\mathbb{R}^n$. Already solving a simple boundary value problem on $(0, 1)$ (the first construction of wavelets on the interval [20] was published in 1993) posed a challenge.

Many steps forward have been made since those pioneering works. In particular *thinking in terms of wavelets* gave birth to some new approaches in the numerical solution of PDEs. The aim of this course is to show some of these new ideas. In particular we want to show how one key property of wavelets (the possibility of writing equivalent norms for the scale of Besov spaces) allows to write down some new methods.

Chapter 1

What is a Wavelet?

Let us start by explaining what we mean by wavelets. There are in the literature many definitions of wavelets and wavelet bases, going from the more strict ones (a wavelet is the dilated and translated version of a *mother wavelet* satisfying a suitable set of properties) to more and more general definitions. The aim of this chapter is to review the classical definition of wavelets for $\mathbb{R}$ and then point out which of its properties can be retained when replacing $\mathbb{R}$ with a generic domain Ω.

1.1 Multiresolution Analysis

We start by introducing the general concept of multiresolution analysis in the univariate case.

Definition 1.1. A *Multiresolution Analysis* (MRA) of $L^2(\mathbb{R})$ is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ verifying:

i) the subspaces are nested: $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

ii) the union of the spaces is dense in $L^2(\mathbb{R})$ and the intersection is null:

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}), \qquad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}; \tag{1.1}$$

iii) there exists a *scaling function* $\varphi \in V_0$ such that $\{\varphi(\cdot - k), k \in \mathbb{Z}\}$ is a Riesz's basis for V_0.

We recall that a set $\{e_k\}$ is a Riesz basis for its linear span in $L^2(\mathbb{R})$ if and only if the functions e_k are linearly independent and the following norm equivalence holds,

$$\Big\| \sum_k c_k e_k \Big\|^2_{L^2(\mathbb{R})} \simeq \sum_k |c_k|^2.$$

Here and in the following we use the notation $A \simeq B$ to signify that there exist positive constants c and C, independent of any relevant parameter, such that $cB \le A \le CB$. Analogously we will use the notation $A \lesssim B$ (resp. $A \gtrsim B$), meaning that $A \le CB$ (resp. $A \ge cB$).

It is not difficult to check that the above properties imply that the set

$$\{\varphi_{j,k} = 2^{j/2}\varphi(2^j \cdot -k),\ k \in \mathbb{Z}\}$$

is a Riesz's basis for V_j, yielding a norm equivalence between the L^2-norm of a function in V_j and the ℓ^2-norm of the sequence of its coefficients with constants independent of j.

The inclusion $V_0 \subset V_1$ implies that the scaling function φ can be expanded in terms of the basis of V_1 through the following *refinement equation*

$$\varphi(x) = \sum_{k\in\mathbb{Z}} h_k \varphi(2x - k) \tag{1.2}$$

with $\{h_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. The function φ is then said to be a *refinable function* and the coefficients h_k are called *refinement coefficients*.

Since $V_j \subset V_{j+1}$ it is not difficult to realize that an approximation f_{j+1} of a function f at level $j+1$ "contains" more information on f than the approximation f_j at level j. As an example, we can consider $f_j = P_j f$, where $P_j : L^2(\mathbb{R}) \to V_j$ denotes the $L^2(\mathbb{R})$-orthogonal projection onto V_j. Remark that $P_{j+1}P_j = P_j$ (a direct consequence of the nestedness of the spaces V_j). Moreover, we have that $P_j P_{j+1} = P_j$: f_{j+1} contains in this case all information needed to retrieve f_j. The idea is now to encode somehow the "loss of information" that we have when projecting f_{j+1} onto V_j. This is done by introducing the complement *wavelet space* W_j. In order to do that, we consider a more general framework, in which P_j is not necessarily the orthogonal projection and which yields the construction of a biorthogonal multiresolution analysis, as specified in the following section.

The Biorthogonal MRA

To be more general, let us start by choosing a sequence of uniformly bounded (not necessarily orthogonal) projectors $P_j : L^2(\mathbb{R}) \to V_j$ verifying the following properties:

$$P_j P_{j+1} = P_j, \tag{1.3}$$

$$P_j(f(\cdot - k2^{-j}))(x) = P_j f(x - k2^{-j}), \tag{1.4}$$

$$P_{j+1} f((2\cdot))(x) = P_j f(2x). \tag{1.5}$$

Remark again that the inclusion $V_j \subset V_{j+1}$ guarantees that $P_{j+1}P_j = P_j$. On the contrary, property (1.3) is not verified by general non-orthogonal projectors and expresses the fact that the approximation $P_j f$ can be derived from $P_{j+1}f$. Equations (1.4) and (1.5) require that the projector P_j respects the translation and dilation invariance properties (i) and (ii) of the MRA.

Since $\{\varphi_{0,k}\}$ is a Riesz's basis for V_0 we have that for $f \in L^2(\mathbb{R})$

$$P_0 f = \sum_k \alpha_k(f)\varphi_{0,k}$$

with $\alpha_k : L^2(\mathbb{R}) \to \mathbb{R}$ linear and continuous. By the Riesz's Representation Theorem, for each k, there exists an element $\tilde{\varphi}_{0,k} \in L^2(\mathbb{R})$ such that

$$\alpha_k(f) = \langle f, \tilde{\varphi}_{0,k}\rangle,$$

where we denote by $\langle\cdot,\cdot\rangle$ the L^2-scalar product. We have the following lemma:

Lemma 1.2. *The set* $\{\tilde{\varphi}_{0,k},\ k \in \mathbb{Z}\}$ *forms a Riesz's basis for the space* $\tilde{V}_0 = P_0^*(L^2(\mathbb{R}))$ *(where* P_0^* *denotes the adjoint of* P_0*). Moreover we have*

$$\tilde{\varphi}_{0,k}(x) = \tilde{\varphi}_{0,0}(x-k). \tag{1.6}$$

Proof. We start by remarking that since $\varphi_{0,n} \in V_0$, we have that

$$\varphi_{0,n} = P_0\varphi_{0,n} = \sum_k \langle \varphi_{0,n}, \tilde{\varphi}_{0,k}\rangle \varphi_{0,k},$$

and this implies

$$\langle \tilde{\varphi}_{0,n}, \varphi_{0,k}\rangle = \delta_{n,k}. \tag{1.7}$$

Remark that (1.7) implies that the functions $\tilde{\varphi}_{0,k}$ are linearly independent. By definition, since $\{\varphi_{0,k}\}$ is a Riesz's basis for V_0 there exist constants A and B such that

$$A\Big(\sum_k |\alpha_k|^2\Big)^{1/2} \le \Big\|\sum_k \alpha_k\varphi_{0,k}\Big\|_{L^2(\mathbb{R})} \le B\Big(\sum_k |\alpha_k|^2\Big)^{1/2}.$$

We have

$$\begin{aligned}
\Big\|\sum_k \xi_k\tilde{\varphi}_{0,k}\Big\|_{L^2(\mathbb{R})} &= \sup_{f\in L^2(\mathbb{R})} \frac{\langle f, \sum_k \xi_k\tilde{\varphi}_{0,k}\rangle}{\|f\|_{L^2(\mathbb{R})}} = \sup_{f\in L^2(\mathbb{R})} \frac{\sum_k \alpha_k(f)\xi_k}{\|f\|_{L^2(\mathbb{R})}} \\
&\lesssim \sup_{f\in L^2(\mathbb{R})} \frac{(\sum_k |\alpha_k(f)|^2)^{1/2}(\sum_k |\xi_k|^2)^{1/2}}{\|f\|_{L^2(\mathbb{R})}} \\
&\lesssim \sup_{f\in L^2(\mathbb{R})} \frac{\|P_0 f\|_{L^2(\mathbb{R})}(\sum_k |\xi_k|^2)^{1/2}}{\|f\|_{L^2(\mathbb{R})}} \lesssim \Big(\sum_k |\xi_k|^2\Big)^{1/2}.
\end{aligned} \tag{1.8}$$