# WAVELETS
## Theory and Applications

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## PREFACE

This volume contains the formal notes of the short course on wavelets conducted by the Institute for Computer Applications in Science and Engineering (ICASE) and NASA Langley Research Center (LaRC) during February 22-26, 1993. The purpose of the short course was to give scientists and engineers a practical understanding of wavelets: their applications as a diagnostic tool and their use as basis functions to solve differential equations. The emphasis was on providing as much as possible the practical knowledge which will enable applied scientists to evaluate objectively how useful these new tools are in relation to their needs.

To this end, each chapter is written with the practical user in mind. Instead of concentrating on the theoretical aspects of wavelets, which can become quite incomprehensible to the uninitiated, the authors strive to bring the subject down to a level where it can possibly be appreciated by the research engineer. However, since the field of wavelets has such a strong theoretical basis, we have included a chapter on the theory of wavelets and operators, in which the subject matter is rather theoretical.

This volume is divided into seven chapters. All the chapters were contributed by the instructors of the shortcourse except the first one, Introduction to Wavelets and their Application to Partial Differential Equations, by Jameson of ICASE.

The objective of Chapter 1 is to introduce the reader to the basic wavelet concepts related to the solution of partial differential equations. Details of wavelet theory and their construction are skipped, and the reader is referred to the other articles in the volume. Techniques discussed include the accuracy of the derivative operator representation in wavelet bases, adaptive methods, and collocation methods. In addition, an introduction to the recent incomplete theory of Amiram Harten is included.

Chapter 2, authored by Strang, introduces the concept of multiresolution spaces which is crucial to the entire field of wavelets. He then discusses the idea of filters, and how wavelet and scaling spaces are related to the notion of high pass and low pass filters, respectively. Next, he discusses the dilation equation which relates two spaces at successive scales. This study is presented very intuitively in the physical and the frequency domain.

Armed with this strong introduction, Tchamitchian in Chapter 3 introduces the theoretical framework from which the various applications of wavelets can be understood. After setting the stage with a historical background and some basic concepts, he explains how to use the modulus and the phase of the wavelet coefficients to extract information from time-dependent signals. This is followed by some theoretical considerations on multiresolution analysis which complements nicely the presentation in Chapter 1. The second half of this chapter is for the more theoretically inclined. Numerical analysts and applied mathematicians are often confronted with the need to invert operators. In Sections 8 and 9, Tchamitchian discusses spaces of operators and explains the circumstances under which they can be efficiently inverted. Finally, a theoretical discussion on wavelet adaptation is presented. Adaptivity plays a very important role in wavelet-based algorithms since the localization of scales in space and time is then exploited fully.

In Chapter 4, Beylkin introduces fast numerical algorithms to perform a variety of useful operations. These include the representation of differential operations in a wavelet basis, the compression of operators, multiplication of operators, and the convolution of operators. The building blocks for these algorithms is the transformation of operators into standard and nonstandard forms. Some practical wavelet-based algorithms are given for standard operations, such as a generalized inverse, the inverse of the second derivative operator, and an algorithm to compute the square root. In preparation for the need to solve nonlinear equations, Beylkin finally considers the representation of nonlinear terms, particularly quadratic nonlinearities using wavelet basis functions.

In Chapter 5, Liandrat discusses the motivation for using wavelets to solve PDE's. After explaining the result of differential operators on wavelets, he comments on the approximation of certain classes of elliptic operators with wavelets. This serves as a starting point from which adaptive algorithms are introduced. Examples based on Burgers equation are presented using both adaptive and nonadaptive methods.

Finally, Chapters 6 and 7 focus on the application of wavelets to the study of experimental signals. In Chapter 6, Liandrat, starting from the continuous wavelet, explains how to relate the wavelet coefficient to the energy of a signal, and then, making use of the redundancy in position and scale offered by wavelets, how to extract local information from a signal, both in scale and time. The algorithms developed are then applied to experimental time histories taken from a rotating disk experiment designed to study transition to turbulence.

Fractals and multifractals are then discussed by Arneodo in Chapter 7. The first half of the chapter is dedicated to a thorough review of fractal and multifractal concepts, fractal measures, and fractal functions. Different techniques are introduced to compute the singularity spectrum of multifractal functions, and their deficiencies are addressed. In the second half of the chapter, Arneodo explains how wavelets can be used to characterize the singularities of functions. They are shown to be well-suited to compute singularity spectrums, both for positive and negative dimension. Wavelets are then used to describe turbulent signals. Then the author's theory of fractal growth phenomena is unfolded and the role of wavelets in extracting the inherent structure in these aggregates is presented. Finally, some thoughts on the inverse fractal problem are advanced.

The editors would like to take the opportunity to thank all the contributors for a job well done. It is a pleasure to acknowledge the assistance of Emily Todd who coordinated the preliminary correspondence for the Short Course as well as the collection and format editing of the typescripts. We are also thankful to Jeff Robbins of Oxford University Press for his cooperation and patience in bringing out this volume.

Gordon Erlebacher
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Leland M. Jameson

# 1

# INTRODUCTION TO WAVELETS AND THEIR APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

L.M. Jameson

## Contents

Erratum to *Wavelets: Theory and Applications*

Page one of this book was unfortunately omitted from the bound edition, and replaced with a halftitle page. We hope this causes no confusion. Page one listed the chapter title and detailed chapter contents for L. M. Jameson's chapter, "Introduction to Wavelets and Their Application to Partial Differential Equations" as shown in the Table of Contents, page ix.

## CONTENTS

# Wavelets

# 1 Introduction and Motivation

This introduction to wavelets is written for the engineer and not for the pure mathematician, and consequently, relies more on intuition and calculus than on functional analysis. That is, technical mathematical language is avoided and the emphasis is placed on how to use wavelets as a tool to help one to get an answer to a physical problem. Of particular interest are the possible applications of wavelets in computational fluid dynamics (CFD). For introductions to wavelet methods which are slightly more theoretical, and consequently more mathematically precise, as well as being in the context of functional analysis, see the chapters by Liandrat and Tchamitchian in this volume.

The term wavelet refers to sets of functions of the form

$$\psi_{ab}(x) = |a|^{-1/2}\psi(\frac{x-b}{a}),$$

normalized by $|a|^{-1/2}$, i.e., sets of functions formed by the dilations, which are controlled by the positive real number $a \in R^+$, and translations which are controlled by the real number $b \in R$, of a single function $\psi(x)$ often named the mother wavelet. Visually, the mother wavelet appears as a local oscillation, or wave, in which most of the energy of the oscillation is located in a narrow region in the physical space. This localization in the physical space limits the localization in the frequency or wavenumber domain due to the uncertainty principle[1]. The dilation parameter $a$ controls the width and rate of this local oscillation and intuitively can be thought of controlling the frequency of $\psi_{ab}(x)$. The translation parameter $b$ simply moves the wavelet throughout the domain.

If the dilation and translation parameters $a$ and $b$ are chosen such that $a = 2^j$ and $b = k2^j$, where $j$ and $k$ are integers, then there exist wavelets $\psi(x)$ such that the set of functions

$$\psi_k^j(x) = 2^{-j/2}\psi(2^{-j}x - k)$$

constitute an orthonormal basis of the space of functions or signals in $L^2(\mathbb{R})$, which have finite energy (Daubechies 1988 and Daubechies 1992), and as above, the two parameters, $j$ and $k$ can be varied for

[1] The uncertainty principle places a lower bound on the product of the variances of a Fourier transform pair.

analysis of local features of a given function. Note that as the wavelet is stretched by increasing $j$, say from $j = 0$ to $j = 1$, that the translation distance is, also, accordingly increased so that a translation of size $2^0 k$ when $j = 0$ becomes a translation of size $2^1 k$ when $j = 1$. These two degrees of freedom, $j$ and $k$, give one the ability to resolve features at a variety of scales by adjusting $j$ and at any location by adjusting $k$. In a Fourier basis, by constrast, the basis functions are a one-parameter family, $(e^{ix})^n$, indexed by the frequency, $n$, and one can effectively analyze global, periodic, smooth features by adjusting this parameter $n$. A Fourier basis is, however, less conducive for the analysis of localized oscillations or structure.

Consider the following examples from CFD where wavelet methods have either already been shown to be effective or appear to be promising areas for future research:

- **Shock Wave Analysis:** In CFD one often encounters discontinuities, or shock waves, either as an initial condition or induced by the nonlinear terms of the governing equations. In the physical space a discontinuous shock wave is highly localized and contains information at infinitly small scales. When viewed in either a Fourier or wavelet transform space, this information is distributed across all the basis functions which cross the shock location. In the language of wavelet analysis, an inner product of the function representing the shock wave with the wavelet $\psi^j_k(x)$ produces relatively large coefficients when the translation parameter $k$ places $\psi^j_k(x)$ near the discontinuity. As the dilation parameter $j$ produces wavelets on increasingly fine scales, one can find the location and intensity of the shock wave for edge detection (Mallat and Zhong 1992). For solution to partial differential equations containing shocks see Liandrat in this volume, Cai and Wang (1993), Bauer and Jameson (1994).

- **Aeroacoustics:** In aeroacoustics, one of the goals is to follow the motion of a localized wave or wavepacket for a long time. Such a localized phenomenon is exactly the type of data which a wavelet basis can compress effectively. In terms of coefficients of the inner product of the data with the wavelets $\psi^j_k(x)$ one would see that the coefficients are near zero away from the oscillation which allows data compression in this region. Near the oscillation, on the other hand, the inner product coefficients would be relatively large and would provide sufficient information to the equations which govern the wave motion. This type of wavelet analysis is quite robust when used as a grid selection mechanism which depends only on the local oscillation content of the functions involved (Carpenter and Jameson, in preparation).

- **Turbulence:** In turbulence, on the other hand, one has small-scale and large scale structures which can appear in any region of the domain, and a wavelet basis, due to existence of basis functions at all scales, can be used to analyze effectively such flows to aid in understanding the interaction of the various scales (Farge 1992) and the chapter by Arneodo in this volume.

- **Combustion:** In combustion it is critical to resolve the reaction zone of a flame front in order to determine proper flame speed. This reaction zone is typically very narrow and it, consequently, requires very fine numerical resolution but only in a small portion of the domain. An adaptive wavelet-based numerical method can locate and resolve this flame front without over-resolving the remainder of the domain. Furthermore, the data compression, or grid refinement, is not explicely in terms of the physics, but depends only on the existence of small-scale and large-scale structure in the functions involved (Jameson, Jackson and Lasseigne 1994).

From the above examples, it is clear that there is a need for a numerical method in CFD which can efficiently work with information which exists at many different scales in different regions of the domain, and a wavelet basis appears to be a very promising candidate for such a method. The following section gives a more structured introduction to Daubechies-based wavelets.

## 2 Definition of Daubechies-Based Wavelets

To define Daubechies-based wavelets (Daubechies 1988), consider the two functions $\phi(x)$, the scaling function, and $\psi(x)$, the wavelet. The scaling function is the solution of the dilation equation,

$$\phi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k \phi(2x-k), \tag{1}$$

where $\phi(x)$ is normalized $\int_{-\infty}^{\infty}\phi(x)dx = 1$, and the wavelet $\psi(x)$ is defined in terms of the scaling function,

$$\psi(x) = \sqrt{2}\sum_{k=0}^{L-1} g_k\phi(2x-k). \tag{2}$$

- Note that in the chapter by Strang following this introduction that the above coefficients $h_k$ and $g_k$ defining the scaling function and the wavelet will be denoted by $c_k$ and $d_k$, respectively. Other inconsistencies in notational use will be noted as they are encountered.

One builds an orthonormal basis from $\phi(x)$ and $\psi(x)$ by dilating and translating to get the following functions:

$$\phi_k^j(x) = 2^{-\frac{j}{2}}\phi(2^{-j}x-k), \tag{3}$$

and

$$\psi_k^j(x) = 2^{-\frac{j}{2}}\psi(2^{-j}x-k), \tag{4}$$

where $j;k \in \mathbb{Z}$. $j$ is the dilation parameter and $k$ is the translation parameter. The coefficients $H = \{h_k\}_{k=0}^{L-1}$ and $G = \{g_k\}_{k=0}^{L-1}$ are related by $g_k = (-1)^k h_{L-k}$ for $k = 0, ..., L-1$. All wavelet properties are specified through the parameters $H$ and $G$. If one's data is defined on a continuous domain such as $f(x)$ where $x \in R$ is a real number then one uses $\phi_k^j(x)$ and $\psi_k^j(x)$ to perform the wavelet analysis. If, on the other hand, one's data is defined on a discrete domain such as $f(i)$ where $i \in \mathbb{Z}$ is an integer then the data is analyzed, or filtered, with the coefficients $H$ and $G$. In either case, the scaling function $\phi(x)$ and its defining coefficients $H$ detect localized low frequency information, i.e., they are low-pass filters (LPF), and the wavelet $\psi(x)$ and its defining coefficients $G$ detect localized high frequency information, i.e., they are high-pass filters (HPF). Specifically, $H$ and $G$ are chosen so that dilations and translations of the wavelet, $\psi_k^j(x)$, form an orthonormal basis of $L^2(\mathbb{R})$ and so that $\psi(x)$ has $M$ vanishing moments which determines the accuracy (Strang 1992). In other words, $\psi_k^j(x)$ will satisfy

$$\delta_{kl}\delta_{jm} = \int_{-\infty}^{\infty}\psi_k^j(x)\psi_l^m(x)dx, \tag{5}$$