Progress in Mathematics

Edited by
J. Coates and
S. Helgason

Classification of Algebraic and Analytic Manifolds

Kenji Ueno, editor

Birkhäuser

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Katata Symposium Proceedings 1982 Sponsored by the Taniguchi Foundation

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1983

Birkhäuser
Boston • Basel • Stuttgart

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Library of Congress Cataloging in Publication Data
Main entry under title:

Classification of algebraic and analytic manifolds.

(Progress in mathematics ; v. 39)
Articles based on the 11th international symposium of the Taniguchi Foundation held at Katata, Japan, July 7-13, 1982.
1. Complex manifolds—Congresses. 2. Surfaces, Algebraic—Congresses. 3. Mappings (Mathematics)—Congresses. I. Ueno, Kenji, 1945- II. Taniguchi Kōgyō Shōreikai. III. Series: Progress in mathematics (Cambridge, Mass.) ; v. 39.

QA331.C565 1983 515 83-15598
ISBN 0-8176-3137-2

CIP-Kurztitelaufnahme der Deutschen Bibliothek

Classification of algebraic and analytical manifolds :
Katata symposium proceedings 1982 / Kenji Ueno, ed.
Sponsored by the Taniguchi Foundation. - Boston ;
Basel ; Stuttgart : Birkhäuser, 1983.
(Progress in mathematics ; Vol. 39)
ISBN 3-7643-3137-2 (Basel, Stuttgart)
ISBN 0-8176-3137-2 (Boston)

NE: Ueno, Kenji [Hrsg.]; GT

ISBN 0-8176-3137-2
ISBN 3-7643-3137-2
Printed in USA

Progress in Mathematics
Vol. 39

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Preface

The 11-th international symposium of the division of mathematics of the Taniguchi Foundation on Classification of Algebraic and Analytic Manifolds was held at Katata, Japan, July 7-13, 1982. The present volume contains 15 articles based on the talks given at Katata. In the symposium much time was spent to discuss open problems related to the classification theory. The present volume also contains the list of open problems with several comments.

We were given generous financial support by the Taniguchi Foundation as well as warm hospitality of Mr. T. Taniguchi. We were also indebted to Professors Y. Akizuki and S. Murakami for organizing the symposium. We would like express our hearty thanks to them and the Taniguchi Foundation.

Kenji Ueno

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CONTENTS

SOME REMARKS ON KÄHLER MANIFOLDS WITH $c_1 = 0$

Arnaud BEAUVILLE

These notes consist mainly of comments and applications of the results of [B]. We first recall the structure theorem for compact Kähler manifolds with $c_1 = 0$. Up to a finite covering, such a manifold splits as a product of irreducible factors of 3 possible types : complex tori, special unitary projective manifolds and Kähler symplectic manifolds. After showing a list of examples we give some applications, mainly to the study of automorphisms. We extend to our manifolds some results of Nikulin on automorphisms of K3 surfaces. We then consider automorphisms of symplectic manifolds which induce the identity in cohomology. Finally we conclude with an example of a birational automorphism (of a projective symplectic manifold) which is not biregular, contrary to a conjecture of Bogomolov.

Parts of this paper grew out from stimulating discussions at the Katata Conference. I wish to express my thanks to the Taniguchi Foundation for making possible such a conference, and to K. Ueno for organizing it so nicely.

1. The Structure Theorem

Let me first set up some terminology. A manifold is always assumed to be connected. By a Kähler manifold I mean a complex manifold which admits at least one Kähler metric.

The structure theorem for manifolds with $c_1 = 0$ goes back, in a weak form, to Calabi [C]. A stronger version was proved by Bogomolov in 1974 [Bo]. Finally the proof by S-T. Yau of the Calabi conjecture made possible to give an easy proof of the strongest possible statement. This fact seems to have been noticed independently by various mathematicians, in particular S. Kobayashi and M.L. Michelsohn [M].

Theorem

Let X be a compact Kähler manifold with $c_1^{\mathbb{R}}(X) = 0$.

1) The universal covering of X is isomorphic to a product $\mathbb{C}^k \times \prod_i V_i \times \prod_j X_j$, where

a) V_i is a simply connected projective manifold, of dimension ≥ 3, with trivial canonical bundle, such that $H^0(V_i, \Omega^p_{V_i}) = 0$ for $0 < p < \dim(V_i)$.

b) X_j is a simply connected compact Kähler manifold, admitting a holomorphic 2-form φ_j which is everywhere non-degenerate (as an alternate form on the holomorphic tangent bundle). Any holomorphic form on X_j is (up to a scalar) a power of φ_j.

This decomposition is unique, up to the order of the V_i's and of the X_j's.

2) There exists a finite étale cover $\tilde{X}$ of X which is isomorphic to a product $T \times \prod_i V_i \times \prod_j X_j$, where T is a complex torus.

Let us give a sketch of the proof, referring to [B]for the details. According to Yau's theorem, X carries a Ricci-flat Kähler metric. The De Rham theorem ([K-N], IX.8) implies that the universal covering of X is isomorphic (as a Kähler manifold) to a product $\mathbb{C}^k \times \prod_i M_i$, where for each i the manifold M_i has irreducible holonomy. Moreover M_i is compact by the Cheeger-Gromoll theorem [C-G]. Since M_i is Ricci-flat, its holonomy group H_i is contained in $SU(m_i)$. The list of holonomy groups given by Berger [Be] leaves only two possibilities for H_i, namely $H_i = SU(m_i)$ and $H_i = Sp(m_i/2)$ (if m_i is even).

We now consider holomorphic forms on M_i. The Bochner principle implies that on a compact Kähler Ricci-flat manifold any holomorphic form is parallel. Therefore the space of holomorphic p-forms on M_i is holomorphic to the space of those p-forms at a given point which are invariant under H_i. From the representation theory of the unitary and symplectic groups one deduces easily that M_i satisfies property a) of the theorem if $H_i = SU(m_i)$ and property b) if $H_i = Sp(m_i/2)$ (in case $H_i = SU(m_i)$ with $m_i \geqslant 3$, we observe that the vanishing of $H^{2,0}$ implies that M_i is projective).

This proves the existence of the decomposition 1). The unicity is deduced easily from the unicity of the De Rham decomposition and the unicity of a Ricci-flat metric in a given cohomology class. Finally 2) follows essentially from the classical Bieberbach theorem.

For obvious reasons, manifolds satisfying property a) will be called special unitary, while those satisfying b) will be called (irreducible) symplectic.

Let us mention some obvious consequences of the theorem. The fundamental group of X is an extension of a finite group by a group $\mathbb{Z}^{2\tilde{q}}$ where $\tilde{q}$ is the maximum irregularity of the finite coverings of X. If $\chi(\mathcal{O}_X)$ is nonzero, then $\tilde{q} = 0$ and $\pi_1(X)$ is finite. In any case the canonical bundle is a torsion element of Pic(X).

The following consequences are perhaps less obvious :

<u>Corollary</u> :

<u>Let</u> X <u>be a compact Kähler manifold with</u> $c_1^{\mathbb{R}}(X) = 0$, <u>of dimension</u> n.

(i) <u>If</u> n <u>is odd</u>, <u>one has</u> $\chi(\mathcal{O}_X) = 0$.

(ii) <u>If</u> $n = 2r$, <u>one has</u> $0 \leqslant \chi(\mathcal{O}_X) \leqslant 2^r$. <u>The equality</u> $\chi(\mathcal{O}_X) = 2^r$ <u>holds if and only if</u> X <u>is a product of</u> K3 <u>surfaces</u>.

(iii) <u>One has</u> $h^{p,0}(X) \leqslant \binom{n}{p}$ <u>for all</u> p. <u>If equality holds for one value of</u> p <u>with</u> $0 < p < n$, <u>then</u> X <u>is a complex torus</u>.

The assertion (i) follows at once from Serre duality. Let $\tilde{X}$ be a finite covering of X which is isomorphic to a product of manifolds M_i

of dimension m_i, which either are complex tori, or satisfy property a) or b). Then

$$0 \leqslant \chi(\mathcal{O}_{M_i}) \quad \frac{m_i}{2} + 1 \leqslant 2^{m_i/2} \quad ,$$

with equality if and only if M_i is a K3 surface. Since

$$\chi(\mathcal{O}_X) \leqslant \chi(\mathcal{O}_X) = \prod_i \chi(\mathcal{O}_{M_i}) \text{ , this implies (ii).}$$

Let us prove (iii). Let T_i be a complex torus of dimension m_i.

One has
$$h^{p,0}(M_i) \leqslant h^{p,0}(T_i) \ ,$$

and equality holds (for $0 < p < m_i$) if and only if M_i is a complex torus. We conclude that

$$h^{p,0}(\tilde{X}) \leqslant h^{p,0}(\Pi\, T_i) = \binom{n}{p} \quad ,$$

with equality (for $0 < p < m_i$) if and only if $\tilde{X}$ is a complex torus. It remains to show that in this last case X also is a complex torus. We can assume that the covering $\tilde{X} \to X$ is Galois ; its Galois group G must act trivially on $H^{p,0}(\tilde{X})$. This implies that any element g of G acts on $H^{1,0}(\tilde{X})$ by multiplication by a p-th root of unity $\lambda(g)$. But then the holomorphic Lefschetz fixed-point formula, applied to g, gives

$$1 - \binom{n}{1} \lambda(g) + \binom{n}{2} \lambda^2(g) + \ldots + (-1)^n = (1-\lambda(g))^n = 0 \ ,$$

hence $\lambda(g) = 1$, which means that G acts on $\tilde{X}$ by translations, so that X is a torus.

2. Examples

a) Special unitary manifolds.

Except K3 surfaces and their products, all usual examples of Kähler manifolds with trivial canonical bundle are special unitary : hypersurfaces of degree (m+2) in $\mathbb{P}^{m+1}$, complete intersections of degrees $(d_1, \ldots, d_r)$ in $\mathbb{P}^n$, with $\Sigma d_i = n+1$; more generally, weighted complete intersections of degrees $(d_1, \ldots, d_r)$ in the twisted projective space

$\mathbb{P}(e_1,\dots,e_n)$ with $\Sigma d_i = \Sigma e_i$. If V is a projective manifold with ample anticanonical bundle (for instance a complex homogeneous space G/P, where G is a semi-simple complex Lie group and P a parabolic subgroup), then any smooth hypersurface $X \in |-K_V|$ is special unitary. More generally if $X_1,\dots,X_r$ are ample divisors in V meeting transversally, such that $\Sigma X_i \equiv -K_V$, then $X = \bigcap_i X_i$ is special unitary, etc ...

Let me give another example which is of a somewhat different nature. For m=3, 4 or 6, let E_m denote the elliptic curve which admits an automorphism of order m. Put $A_m = (E_m)^m$. The group μ_m of m-th roots of unity acts diagonally on A_m, with a finite number of fixed points. By blowing-up these points, we obtain a manifold $\hat{A}_m$ on which the group μ_m acts in such a way that the locus of fixed points of a generator is a smooth divisor. Therefore the manifold $X_m = \hat{A}_m/\mu_m$ is smooth. One checks easily that X_m is simply connected and that its canonical bundle is trivial. For $p \neq q$ one has

$$H^{p,q}(X_m) = H^{p,q}(\hat{A}_m)^{inv} = H^{p,q}(A_m)^{inv}$$

(here the sign = means "canonically isomorphic"). Let $V = H^{1,0}(A_m)$; the group μ_m acts on V by multiplication. Then we have $H^{p,q}(X_m) = (\Lambda^p V \otimes \Lambda^q \bar{V})^{inv}$, so $H^{p,q}(X_m) = 0$ for $p \neq 0,m$ and $p \neq q$. This implies that the manifolds X_m are special unitary. They have some interesting properties : in particular they are rigid, since

$$H^1(X_m,T_{X_m}) = H^1(X_m,\Omega^{m-1}_{X_m}) = H^{m-1,1} = 0 .$$

b) Symplectic manifolds

A symplectic structure on a complex manifold X is a holomorphic 2-form on X which is everywhere non-degenerate. The existence of such a structure implies that X is even-dimensional and has trivial canonical bundle. It follows from the structure theorem that a compact Kähler manifold is symplectic irreducible (in the sense of the theorem) iff it is simply connected and admits a unique symplectic structure (up to a scalar).

Let S be a compact complex surface. We denote by $S^{(r)}$ the r-th symmetric product of S (quotient of S^r by the symmetric group $\mathfrak{S}_r$) and by $\pi: S^r \to S^{(r)}$ the quotient map. The (singular) variety $S^{(r)}$ parametrizes effective 0-cycles of degree r on S. Let $S^{[r]}$ be the Douady Space of 0-dimensional subspaces $Z \subset S$ with $\ell g(\mathcal{O}_Z) = r$.

Consider the natural map $\varepsilon: S^{[r]} \to S^{(r)}$ which associates to a finite subspace the corresponding 0-cycle. Let D be the diagonal of $S^{(r)}$ (locus of cycles $2p_1 + \ldots + p_{r-1}$), and put $E = \varepsilon^{-1}(D)$. It is clear that $\varepsilon: S^{[r]} - E \to S^{(r)} - D$ is an isomorphism, so ε is a bimeromorphic morphism. Fogarty has proved that $S^{[r]}$ is smooth, so that ε is a resolution of the singularities of $S^{(r)}$. Note that the exceptional divisor E is irreducible (Iarrobino).

Proposition 1:

Let S be a generic K3 surface. Then $S^{[r]}$ is a Kähler symplectic manifold, irreducible, of dimension $2r$.

Here the word "generic" means that S is allowed to vary in an open dense subset of the coarse moduli space of Kähler K3 surfaces, containing the projective ones (I have to make this rather unpleasant restriction only because I don't know how to prove that $S^{[r]}$ is Kähler for every S). For $r = 2$, this example has been first noticed by A. Fujiki (see [F2]).

Again I refer to [B] for a complete proof ; I just want to sketch how one gets the symplectic structure on $S^{[r]}$. Let S^r_* denote the set of r-uples $(x_1, \ldots, x_r)$ with at most two x_i's equal. Put $S^{(r)}_* = \pi(S^r_*)$ and $S^{[r]}_* = \varepsilon^{-1}(S^{(r)})$. Then the map $\varepsilon : S^{[r]}_* \to S^{(r)}_*$ is easy to understand. Since a subspace with associated cycle $2p$ is given by a point of $\mathbb{P}(T_p(S))$, it is easily checked that ε is just the blowing-up of $D \cap S^{(r)}_*$ in $S^{(r)}_*$. More precisely, let $\Delta = \pi^{-1}(D)$ be the diagonal of S^r ; note that $\Delta \cap S^r_*$ is smooth of codimension 2 in S^r_*. If $\eta: B_\Delta(S^r_*) \to S^r_*$ denotes the blowing-up of S^r_* along Δ, then we get a commutative diagram

$$\begin{array}{ccc} B_\Delta(S^r_*) & \xrightarrow{\ \eta\ } & S^r_* \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\pi} \\ S^{[r]}_* & \xrightarrow{\ \varepsilon\ } & S^{(r)}_* \end{array} \quad ,$$

where ρ is a Galois covering with group $\mathfrak{S}_r$, ramified simply along the exceptional divisor E' of η.

From a nonzero 2-form on S we deduce a symplectic structure ω on S^r. The form $\eta^*\omega$ is invariant under $\mathfrak{S}_r$, thus descends to a holomorphic 2-form φ on $S^{[r]}_*$ with $\rho^*\varphi = \eta^*\omega$. We have

$$\rho^*\mathrm{div}(\varphi^r) = \mathrm{div}(\rho^*\varphi^r) - E' = \mathrm{div}(\eta^*\omega^r) - E' = 0 \ ,$$

hence $\mathrm{div}(\varphi^r)=0$, which implies that φ is a symplectic structure on $S^{[r]}_*$. Now since E is irreducible, $S^{[r]} - S^{[r]}_*$ is of codimension $\geqslant 2$ in $S^{[r]}$; so by Hartogs' theorem φ extends to a holomorphic 2-form $\tilde{\varphi}$ on $S^{[r]}$. The divisor of $\tilde{\varphi}^r$, which should be contained in $S^{[r]} - S^{[r]}_*$ is zero, which means that $\tilde{\varphi}$ is a symplectic structure on $S^{[r]}$.

Now let A be a 2-dimensional complex torus. The manifold $A^{[r]}$ is again symplectic, but not simply connected. Let $s : A^{(r)} \to A$ be the sum map (defined by $s([a_1]+\ldots+[a_r]) = \sum_i a_i$). By composition with ε, we obtain a morphism $S : A^{[r]} \to A$.

The group A acts on $A^{[r]}$ by translations. Let us also consider its action on A given by $(\alpha,a) \mapsto a + r\alpha$. Then the map S is equivariant with respect to these actions, so it is smooth and has isomorphic fibres. We put $K_{r-1} = S^{-1}(0)$. In the same way as prop. 1, we prove in [B] :

Proposition 2 :

For A *a generic 2-dimensional complex torus, the manifold* K_r *is Kähler symplectic, irreducible, of dimension* 2r.

The manifold K_1 is simply the Kummer surface associated to A. So the manifolds $S^{[r]}$ appear as natural generalizations of K3 surfaces, while K_r seems to generalize Kummer surfaces. Note however that for $r \geqslant 2$ the manifolds $S^{[r]}$ and K_r are not isomorphic.

It turns out that the manifolds $S^{[r]}$ (resp. K_r) have more deformations than those coming from deformations of S(resp. A) : these deformations furnish new (although not very explicit) examples of Kähler symplectic manifolds. At the moment I know no other types of such manifolds.

3. *Split coverings*

In this section we want to state more precisely the assertion 2) of the structure theorem, and in particular give a corresponding assertion of unicity. This will follow from general remarks about Kähler manifolds which are covered by a product of a complex torus and a compact simply connected manifold.

Lemma :

Let T *be a complex torus and* S *be a compact Kähler manifold with* $b_1(S) = 0$. *Then any automorphism* u *of* $T \times S$ *is of the form* (v,w), *with* $v \in Aut(T)$ *and* $w \in Aut(S)$.

Since the projection $T \times S \to T$ is the Albanese map of $T \times S$, there is a commutative diagram

$$\begin{array}{ccc} T \times S & \xrightarrow{\quad u \quad} & T \times S \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad v \quad} & T \end{array}$$

This implies the existence of a map w of T into the complex Lie group Aut(S) such that

$$u(t,s) = (v(t), w_t(s)) \quad \text{for} \quad t \in T \quad \text{and} \quad s \in S.$$

Now the map $t \mapsto w_t w_o^{-1}$ gives an action of T on S. Since S is Kähler with $b_1(S) = 0$, it is known that such an action is necessary trivial (see e.g. [F1]), which implies the lemma.

In what follows, a covering is always assumed to be étale. We'll say for short that a compact manifold $\widetilde{X}$ is split if it is isomorphic to the product of a torus and a simply connected (compact) manifold. Let X be a compact manifold ; we'll say that a finite covering $\widetilde{X} \to X$ is split if the manifold X is split. Finally we'll say that a split covering $T \times S \to X$ is minimal if it is Galois and if its Galois group does not contain any element of the form $(\tau, 1_S)$, where τ is a translation of the torus T.

Proposition 3 :

Let X be a compact complex manifold which admits a finite split covering. Then there exists a minimal split covering $\tau : T \times S \to X$, unique (up to a non-unique automorphism). Any split covering of X factors through π .

We first observe that every finite covering of a split manifold is split ; therefore there exists a split covering $\pi : T \times S \to X$ which is Galois. Let G be its Galois group, and let K be the subgroup of G consisting of automorphisms $(\tau, 1_S)$, where τ is a translation. Put $T = \widetilde{T}/K$. Then K is a normal subgroup of G (by the lemma) and the covering $\pi : T \times S \to X$ deduced from $\widetilde{\pi}$ is Galois with Galois group G/K, hence is minimal. Let $\pi' : T' \times S' \to X$ be another split covering. Then there exists a Galois covering π'' of X, with Galois