# PRINCIPLES OF DYNAMICS

Second Edition

Donald T. Greenwood

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*second edition*

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# PREFACE

We all have witnessed the regular placing into orbit of mechanically complex satellites with moving parts and appendages. On earth, the production line has been changed toward an increasing usage of intricate robotic devices involving many interconnected bodies. These developments serve to emphasize the need for an understanding of dynamics which goes beyond the elementary level. The aim and purpose of this textbook is to present general theory and illustrative examples, as well as homework problems, in such a manner that the student may attain a real comprehension of the subject at the intermediate level.

It is assumed that the students using this text will have the academic maturity of first-year graduate students or of well-prepared undergraduate seniors. The presentation of the material favors a problem-oriented course which emphasizes the ability to combine theories from the various chapters and to use differential equations in the solution of problems.

An attempt has been made to present the basic theory in a very general manner by applying it first to systems of particles and then to idealized bodies such as rods and disks, and finally to more general rigid bodies and systems of rigid bodies. Consequently, the examples and problems are general in nature, for the most part, and are not slanted toward any particular area of application.

The introductory chapter reviews some of the basic concepts of Newtonian mechanics. There is a discussion of units and their definitions, with the SI system of units being adopted. There is also a first elementary presentation of d'Alembert's principle.

One of the principal sources of difficulty for students of dynamics lies in the subject of kinematics. Therefore, the kinematical foundations of particle motion are discussed rather thoroughly in Chapter 2. Motion in a plane and also general three-dimensional motion are included. Particular attention is given to rotating reference frames and to vector derivatives relative to these frames.

Chapters 3 and 4 are devoted to a general vectorial development of the dynamics of a single particle and of systems of particles. These chapters form the theoretical base upon which much of later developments depends.

Orbital motion is the subject of Chapter 5. The discussion is almost entirely limited to motion in an inverse-square gravitational field. In addition to the derivation of orbital trajectories, some attention is given to the time of flight, the determination of the orbital elements, and to elementary perturbation theory.

Beginning with Chapter 6, extensive use is made of the Lagrangian method of formulating the equations of motion. Because of their great importance in the theoretical development of analytical dynamics, the ideas of virtual displacements and virtual work are carefully presented. The mathematical description of constrained systems is discussed, and the explicit forms of the equations of motion for holonomic and nonholonomic systems are given.

Chapter 7 is concerned primarily with the kinematics and planar dynamics of rigid body motion. Matrix and dyadic notations are introduced in the context of the rotational inertial properties of rigid bodies and the associated eigenvalue problem. The rotation matrix and Eulerian angles are also discussed.

The three-dimensional rotational dynamics of one or more rigid bodies is the subject of Chapter 8. It is in this chapter that many of the theories developed previously are combined in the analysis of relatively complex dynamical motions. The free and forced motions of rigid bodies are discussed. A general form of d'Alembert's principle is used to obtain the equations of motion for systems of rigid bodies.

The final chapter is concerned with vibration theory. The associated eigenvalue problem is applied to linear, or linearized, systems having $n$ degrees of freedom. Other topics such as Rayleigh's principle, the use of symmetry, and the free and forced vibrations of damped systems are also included.

The material contained in this text can be covered in about four semester hours, assuming that part of the class time is allotted to problem discussions. For a course of three semester hours, all of Chapter 5 except perturbation theory and most of Chapter 9 can be omitted without seriously affecting the continuity of the presentation.

In conclusion, I wish to thank my wife who typed the manuscript, helped with the proofreading, and performed various other chores in the process of accomplishing this revision.

*Ann Arbor, Michigan* *Donald T. Greenwood*

# CONTENTS

# 1

# INTRODUCTORY CONCEPTS

The science of *mechanics* is concerned with the study of the interactions of material bodies. *Dynamics* is that branch of mechanics which consists of the study of the *motions* of interacting bodies and the description of these motions in terms of postulated laws.

In this book we shall concentrate on the dynamical aspects of *Newtonian* or *classical nonrelativistic* mechanics. By omitting quantum mechanics, we eliminate the study of the interactions of elementary particles on the atomic or nuclear scale. Further, by omitting relativistic effects, we eliminate from consideration those interactions involving relative speeds approaching the velocity of light, whether they occur on an atomic or on a cosmical scale. Nor shall we consider the very large systems studied by astronomers and cosmologists, involving questions of long-range gravitation and the curvature of space.

Nevertheless, over a broad range of system dimensions and velocities, Newtonian mechanics is found to be in excellent agreement with observation. It is remarkable that three centuries ago, Newton, aided by the discoveries of Galileo and other predecessors, was able to state these basic laws of motion and the law of gravitation in essentially the same form as they are used at present. Upon this basis, using the mathematical and physical discoveries and notational improvements of later investigators, we shall present a modern version of classical dynamics. In the process, we shall employ two general approaches, namely, the *vectorial dynamics* of Newton's laws and the *analytical dynamics* exemplified by Lagrange's equations.

## 1–1 ELEMENTS OF VECTOR ANALYSIS

### Scalars, Vectors, and Tensors

Newtonian mechanics is, to a considerable extent, vectorial in nature. Its basic equation relates the applied force and the acceleration (both vector quantities) in terms of a scalar constant of proportionality called *the mass*. In contrast to Newton's *vectorial* approach, Euler, Lagrange, and Hamilton later emphasized the *analytical* or algebraic approach in which the differential equations of motion are obtained by performing certain operations on a scalar function, thereby simplifying the analysis in some respects. Our approach to the subject will be vectorial for the most part, although some of the insights and procedures of analytical mechanics will also be used.

Because vector operations are so important in the solution of dynamical problems, we shall review briefly a few of the basic vector operations. First, however, let us distinguish among scalars, vectors, and other tensors of higher rank.

A *scalar* quantity is expressible as a single, real number. Common examples of scalar quantities are mass, energy, temperature, and time.

A quantity having direction as well as magnitude is called a *vector*. In addition, vectors must have certain transformation properties. For example, vector magnitudes are unchanged after a rotation of axes. Common vector quantities are force, moment, velocity, and acceleration. If one thinks of a vector quantity existing in a three-dimensional space, the essential characteristics can be expressed geometrically by an arrow or a directed line segment of proper magnitude and direction in that space. But the vector can be expressed equally well by a group of three real numbers corresponding to the components of the vector with respect to some frame of reference, for example, a set of Cartesian axes. If one writes the numbers in a systematic fashion, such as in a column, then one can develop certain conventions which relate the position in the column to a given component of the vector. This concept can be extended readily to mathematical spaces with more than three dimensions. Thus, one can represent a vector in an $n$-dimensional space by a column of $n$ numbers.

So far, we have seen that a scalar can be expressed as a single number and that a vector can be expressed as a column of numbers, that is, as a one-dimensional array of numbers. Scalars and vectors are each special cases of *tensors*. Scalars are classed as tensors of rank zero, whereas vectors are tensors of rank one. In a similar fashion, a tensor of rank two is expressible as a two-dimensional array of numbers; a tensor of rank three is expressible as a three-dimensional array of numbers, and so on. Note, however, that an array must also have certain transformation properties to be called a tensor. An example of a tensor of rank two is the inertia tensor which expresses the essential features of the distribution of mass in a rigid body, as it affects the rotational motion.

We shall have no occasion to use tensors of rank higher than two; hence no more than a two-dimensional array of numbers will be needed to express the quantities encountered. This circumstance enables us to use matrix notation, where convenient, rather than the more general but less familiar tensor notation.

For the most part, we shall be considering motions which can be described mathematically using a space of no more than three dimensions; that is, each matrix or array will have no more than three rows or columns and each vector will have no more than three components. An exception will be found in the study of vibration theory in Chapter 9 where we shall consider eigenvectors in a multidimensional space.

### *Types of Vectors*

Considering the geometrical interpretation of a vector as a directed line segment, it is important to recall that its essential features include *magnitude* and *direction,* but *not location.* This is not to imply that the location of a vector quantity, such as a force, is irrelevant in a physical sense. The location or point of application can be very important, and this will be reflected in the details of the mathematical formulation; for example, in the evaluation of the coefficients in the equations of motion. Nevertheless, the rules for the mathematical manipulation of vectors do not involve location; therefore, from the mathematical point of view, the only quantities of interest are magnitude and direction.

But from the *physical* point of view, vector quantities can be classified into three types, namely, *free vectors, sliding vectors,* and *bound vectors.* A vector quantity having the previously discussed characteristics of magnitude and direction, but no specified location or point of application, is known as a *free vector.* An example of a free vector is the translational velocity of a nonrotating body, this vector specifying the velocity of any point in the body. Another example is a force vector when considering its effect upon translational motion.

On the other hand, when one considers the effect of a force on the rotational motion of a rigid body, not only the magnitude and direction of the force, but also its line of action is important. In this case, the moment acting on the body depends upon the line of action of the force, but is independent of the precise point of application along that line. A vector of this sort is known as a *sliding vector.*

The third type of vector is the *bound vector.* In this case, the magnitude, direction, and point of application are specified. An example of a bound vector is a force acting on an elastic body, the elastic deformation being dependent upon the exact location of the force along its line of action.

Note again that all mathematical operations with vectors involve only their free vector properties of magnitude and direction.

**Equality of Vectors.** We shall use boldface type to indicate a vector quantity. For example, $\mathbf{A}$ is a vector of magnitude $A$, where $A$ is a scalar.

Two vectors $\mathbf{A}$ and $\mathbf{B}$ are equal if $\mathbf{A}$ and $\mathbf{B}$ have the same magnitude and direction, that is, if they are represented by parallel line segments of equal length which are directed in the same sense. It can be seen that the translation of either $\mathbf{A}$ or $\mathbf{B}$, or both, does not alter the equality since they are considered as free vectors.

**Unit Vectors.** If a positive scalar and a vector are multiplied together (in either order), the result is another vector having the same direction, but whose magnitude is multiplied by the scalar factor. Conversely, if a vector is multiplied by a negative scalar, the direction of the resulting vector is reversed, but the magnitude is again multiplied by a factor equal to the magnitude of the scalar. Thus one can always think of a given vector as the product of a scalar magnitude and a vector of unit length which designates its direction. We can write

$$\mathbf{A} = A\mathbf{e}_A \tag{1–1}$$

where the scalar factor $A$ specifies the magnitude of $\mathbf{A}$ and the *unit vector* $\mathbf{e}_A$ shows its direction (Fig. 1-1).

**Addition of Vectors.** The vectors $\mathbf{A}$ and $\mathbf{B}$ can be added as shown in Fig. 1-2 to give the *resultant vector* $\mathbf{C}$. To add $\mathbf{B}$ to $\mathbf{A}$, translate $\mathbf{B}$ until its origin coincides with the terminus or arrow of $\mathbf{A}$. The vector sum is indicated by the line directed from the origin of $\mathbf{A}$ to the arrow of $\mathbf{B}$. It can be seen that

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1–2}$$

since, for either order of addition, the vector $\mathbf{C}$ is the same diagonal of the parallelogram formed by using $\mathbf{A}$ and $\mathbf{B}$ as sides. This is the *parallelogram rule of vector addition.* Since the order of the addition of two vectors is unimportant, vector addition is said to be *commutative.*

This procedure can be extended to find the sum of more than two vectors. For example, a third vector $\mathbf{D}$ can be added to the vector $\mathbf{C}$ obtained previously, giving the resultant vector $\mathbf{E}$. From Fig. 1-3, we see that

$$\mathbf{E} = \mathbf{C} + \mathbf{D} = (\mathbf{A} + \mathbf{B}) + \mathbf{D} \tag{1–3}$$

But we need not have grouped the vectors in this way. Referring again to Fig. 1-3, we see that

$$\mathbf{E} = \mathbf{A} + (\mathbf{B} + \mathbf{D}) \tag{1–4}$$

From Eqs. (1–3) and (1–4) we obtain

$$(\mathbf{A} + \mathbf{B}) + \mathbf{D} = \mathbf{A} + (\mathbf{B} + \mathbf{D}) = \mathbf{A} + \mathbf{B} + \mathbf{D} \tag{1–5}$$

illustrating that vector addition is *associative.*

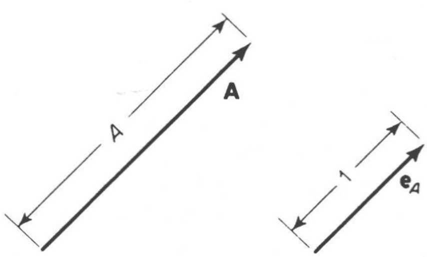


**Figure 1-1** A vector and its corresponding unit vector.

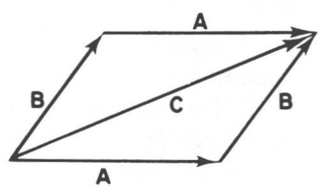


**Figure 1-2** The parallelogram rule of vector addition.

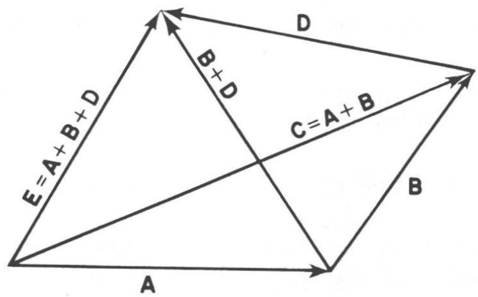


**Figure 1-3** The polygon rule of vector addition.

Because of the commutative and associative properties of vector addition, we can dispense with the parentheses in a series of additions and perform the additions in any order. Furthermore, using the graphical procedure of Fig. 1-3, we see that the resultant vector **E** is drawn from the origin of the first vector **A** to the terminus of the last vector **D**, thus closing the polygon. This generalization of the parallelogram rule is termed the *polygon rule of vector addition.* A similar procedure applies for the case where all vectors do not lie in the same plane.

It is important to note that certain physical quantities that are apparently vectorial in nature do not qualify as true vectors in the sense that the usual rules for vector operations do not apply to them. For example, a finite rotational displacement of a rigid body is not a true vector quantity because the order of successive rotations is important, and therefore it does not follow the commutative property of vector addition. Further discussion of this topic will be found in Chapter 7.

### *Components of a Vector*

If a given vector **A** is equal to the sum of several vectors with differing directions, these vectors can be considered as *component vectors* of **A**. Since component vectors defined in this way are not unique, it is the usual practice in the case of a three-dimensional space to specify three directions along which the component vectors must lie. These directions are indicated by three linearly independent unit vectors, that is, a set of unit vectors such that none can be expressed as a linear combination of the others.

Suppose we choose the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ with which to express the given vector **A**. Then we can write

$$\mathbf{A} = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3 \tag{1–6}$$

where the scalar coefficients $A_1$, $A_2$, and $A_3$ are now determined uniquely. $A_1$, $A_2$, and $A_3$ are known as the *scalar components,* or simply the *components,* of the vector **A** in the given directions.

If another vector **B** is expressed in terms of the same set of unit vectors, for example,

$$\mathbf{B} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3 \tag{1–7}$$

then the components of the vector sum of **A** and **B** are just the sums of the corresponding components.

$$\mathbf{A} + \mathbf{B} = (A_1 + B_1)\mathbf{e}_1 + (A_2 + B_2)\mathbf{e}_2 + (A_3 + B_3)\mathbf{e}_3 \tag{1–8}$$

This result applies, whether or not $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form an orthogonal triad of unit vectors.

Now consider a case where the unit vectors are mutually orthogonal, as in the Cartesian coordinate system of Fig. 1-4. The vector **A** can be expressed in terms of the scalar components $A_x$, $A_y$, and $A_z$, that is, it can be *resolved* as follows:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{1–9}$$

where **i**, **j**, and **k** are unit vectors in the directions of the positive $x$, $y$, and $z$ axes, respectively.

From Fig. 1-4, it can be seen that the component vectors $A_x\mathbf{i}$, $A_y\mathbf{j}$, and $A_z\mathbf{k}$ form the edges of a rectangular parallelepiped whose diagonal is the vector **A**. A similar situation occurs for the case of nonorthogonal or *skewed* unit vectors, except that the parallelepiped is no longer rectangular. Nevertheless, a vector along a diagonal of the parallelepiped has its components represented by edge lengths. In this geometrical construction, we are dealing with free vectors, and it is customary to place the origins of the vector **A** and the unit vectors at the origin of the coordinate system.

It is important to note that, for an *orthogonal* coordinate system, the components of a vector are identical with the orthogonal projections of the given vector onto the coordinate axes. For the case of a skewed coordinate system, however, the scalar components are *not* equal, in general, to the corresponding orthogonal projections. This distinction will be important in the discussions of Chapter 8 concerning the analysis of rigid body rotation by means of Eulerian angles; for, in this case, a skewed system of unit vectors is used.

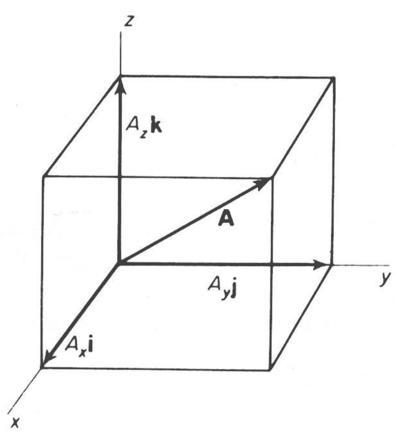


**Figure 1-4** The components of a vector in a Cartesian coordinate system.

### *Scalar Product*

Consider the two vectors **A** and **B** shown in Fig. 1-5. The *scalar product* or *dot product* is

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \tag{1–10}$$

Since the cosine function is an even function, it can be seen that

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1–11}$$

implying that the scalar multiplication of vectors is commutative.

The scalar product can also be considered as the product of the magnitude of one vector and the orthogonal projection of the second vector upon it. Now, it can be seen from Fig. 1-6 that the sum of the projections of vectors **A** and **B** onto a third vector **C** is equal to the projection of **A** + **B** onto **C**. Therefore, noting that the multiplication of scalars is distributive, we obtain

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} \tag{1–12}$$

Thus, the *distributive* property applies to the scalar product of vectors.

Now consider the dot product of two vectors **A** and **B**, each of which is expressed in terms of a given set of unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. From Eqs. (1–6) and (1–7), we obtain

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} = {} & A_1B_1 + A_2B_2 + A_3B_3 + (A_1B_2 + A_2B_1)\mathbf{e}_1 \cdot \mathbf{e}_2 \\ & + (A_1B_3 + A_3B_1)\mathbf{e}_1 \cdot \mathbf{e}_3 + (A_2B_3 + A_3B_2)\mathbf{e}_2 \cdot \mathbf{e}_3 \end{aligned} \tag{1–13}$$

For the common case where the unit vectors form an orthogonal triad, the terms involving dot products of different unit vectors are all zero. For this case, we see from Eq. (1–13) that

$$\mathbf{A} \cdot \mathbf{B} = A_1B_1 + A_2B_2 + A_3B_3 \tag{1–14}$$

### *Vector Product*

Referring again to Fig. 1-5, we define the *vector product* or *cross product* as follows:

$$\mathbf{A} \times \mathbf{B} = AB \sin \theta \, \mathbf{k} \tag{1–15}$$

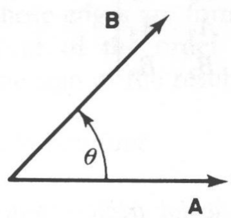


**Figure 1-5** Multiplication of two vectors.

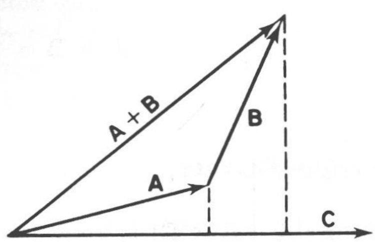


**Figure 1-6** The distributive law for the dot product.

where **k** is a unit vector perpendicular to, and out of, the page. In general, the direction of **k** is found by the right-hand rule, that is, it is perpendicular to the plane of **A** and **B** and positive in the direction of advance of a right-hand screw as it rotates in the sense that carries the first vector **A** into the second vector **B**. The angle of this rotation is $\theta$. It is customary, but not necessary, to limit $\theta$ to the range $0 \leq \theta \leq \pi$.

Using the right-hand rule, it can be seen that

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1–16}$$

indicating that the vector product is *not commutative.* On the other hand, it can be shown that the vector product obeys the *distributive law,* that is,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{1–17}$$

Using the distributive law, we can evaluate the vector product $\mathbf{A} \times \mathbf{B}$ in terms of the Cartesian components of each. Thus,

$$\mathbf{A} \times \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \tag{1–18}$$

But

$$\begin{aligned} \mathbf{i} \times \mathbf{i} &= \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \\ \mathbf{i} \times \mathbf{j} &= -\mathbf{j} \times \mathbf{i} = \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= -\mathbf{k} \times \mathbf{j} = \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= -\mathbf{i} \times \mathbf{k} = \mathbf{j} \end{aligned} \tag{1–19}$$

and therefore,

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \tag{1–20}$$

This result can be expressed more concisely as the following determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1–21}$$

In general, if the sequence $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ forms a right-handed set of mutually orthogonal unit vectors, then the vector product can be expressed in terms of the corresponding components as follows:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{1–22}$$

### *Scalar Triple Product*

The product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is known as the *scalar triple product.* Looking at Fig. 1-7, we see that $\mathbf{B} \times \mathbf{C}$ is a vector whose magnitude is equal to the area of a parallelogram having **B** and **C** as sides and whose direction is perpendicular to the