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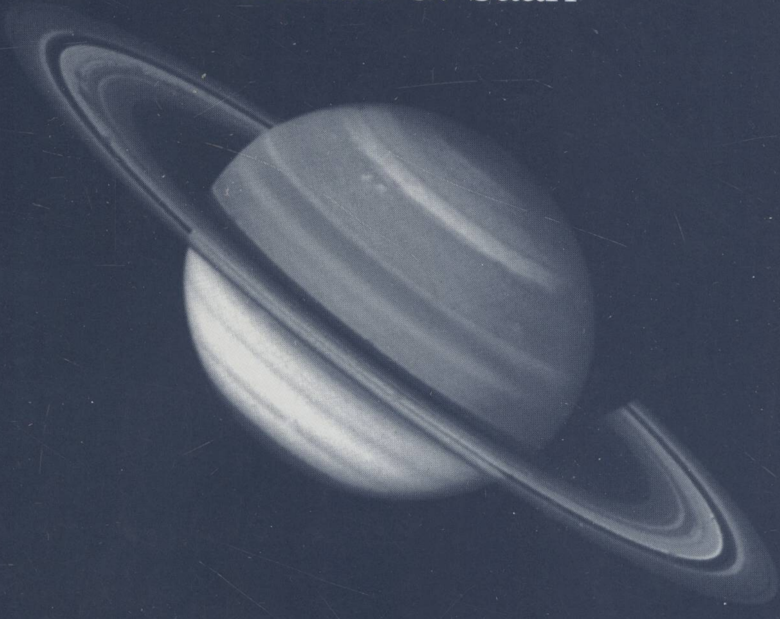
Number 104

Collisions, Rings, and Other Newtonian *N*-Body Problems

Donald G. Saari



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For two great sons-in-law

Adrian Duffin and Erik Sieberg,

and all of my "N-body" students represented by the first and the obvious,

Neal Hulkower and Zhihong (Jeff) Xia

Preface

This book is the written version of my Conference Board of Mathematical Sciences (CBMS) lectures presented during the week of June 10, 2002, at Eastern Illinois University in Charleston, Illinois. The ten lectures centered on my first and persistent academic love—the Newtonian N-body problem.

While some experts actively participated in the sessions, this conference fully lived up to the intent of the CBMS series in that most of the attendees were graduate students, new-comers to the field, or curious mathematicians wishing to learn something about this fascinating topic. Accordingly, the goals of the lectures quickly changed from a technical presentation appropriate primarily for "experts," to presentations now intended to introduce everyone to the basic structure of N-body systems, to identify certain persistent research themes, and, hopefully, to recruit active participants to this fascinating research area. As such, during each lecture several unsolved research problems were described: some of them are included here.

The new goals for the lectures changed the nature, content, expository tone, and even the subject matter to make the presentations more responsive to the specific interests of the participants while addressing their many questions, For instance, I included more introductory material than originally planned: in retrospect, this was an excellent addition.

The content and approach of this book mimic the changed goals of the lectures; e.g., in addition to new material, you will find discussions intended to develop intuition, introductory material, occasional anecdotes, and descriptions of open problems. To provide cohesion for each chapter, some of the material revolves about unsolved research problems—where the motivating role of the problem may be of more value than the actual problem. In Chap. 1, for instance, much of the discussion is intended to lead to an unresolved issue about the weird dynamics exhibited in the F-ring of Saturn. In Chap. 2, the discussion is tied together via a conjecture involving the diameter of the N-body system. In Chap. 3, the unifying problems involve the important issue of finding certain N-body configurations, which leads to a discussion of the rings of Saturn. In Chap. 4, the issue involves collisions. The concluding Chap. 5 discusses the likelihood of "bad things happening." Everyone, from novices to experts, will find something new.

Some results are new, while others have been presented earlier (e.g., at colloquia, Oberwolfach meetings—particularly several during the 1970s—Midwest Dynamical Systems meetings, a 1983 month long mini-course given in Receife, Brazil while visiting Hildeberto Cabral, in a series of lectures in Paris over 1985-87 hosted by Michael Herman, several informal lectures during 1989 in Barcelona hosted by Jaume Llibre, etc.,) and even advertised as "will appear" in fully intended but never completed papers. In other words, many of these results have not been previously published. As most authors of a book quickly discover, the hard part is not to decide what to include, but what to exclude—particularly if a book is to be eventually completed. (Some of the excluded material probably will appear in [90].)

Other results described in this book come from my earlier papers. The particular journals that published these papers are implicitly acknowledged and thanked via the references. But my expository paper [88] *"A visit to the Newtonian n-body via elementary complex variables"* is extensively used to provide structure and motivation for a couple of the chapters, particularly the introductory one, so I want to explicitly thank the MAA for their permission to use it in this manner.

My deep thanks and appreciation go to Patrick Coulton, the chair for this particular CBMS conference, and my long-time friend Gregory Galprin for inviting me to be the CBMS lecturer and for their efforts to assemble a successful CBMS application. I also thank them for their full and active participation in all lectures and extra sessions that they helped to organize, and for everything they did to make the stay so enjoyable for all of us. I want to thank all of the participants for keeping the workshop sessions so lively! My thanks to the Mathematics Department at Eastern Illinois University for their gracious hospitality. My thanks to Ron Rosier and the CBMS for their program that makes these kinds of lectures possible. Thanks to Neal Hulkower: twice at Northwestern he took my year long course on the Newtonian N-body problem (the first in 1969-70), and he still had both sets of lecture notes! His notes proved to be useful in recovering some of my earlier results and arguments. Also thanks to another student (but I do not recall who it was) who gave me a copy of his notes many years ago.

Irvine, California
January, 2005

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Contents

Chapter 1

Introduction

Simply stated, the "Newtonian N-body problem" is the mathematical study of how heavily bodies move in settings where the dynamics are dictated by Newton's law of motion. In practical terms, this area now includes just about any dynamical system that even remotely resembles Newton's law.

Beyond the insight the subject provides for understanding astronomical issues, the Newtonian N-body problem has historically served as a source of mathematical discovery and new problems. The purpose of this book is to introduce the reader to a selective portion of issues about the Newtonian N-body problem while outlining and describing some open problems.[1]

1.1 Mars

How do the heavenly bodies move? A quick introduction can be provided by using elementary complex variables to describe some simple orbits. The ultimate purpose of this exercise is to show how surprising levels of complexity can arise even in particularly "nice" and "well behaved" settings. Later in this chapter, these orbits are used to describe and motivate an open research problem.

Start with a mystery that most surely bothered generations of school kids: it most certainly troubled me when I was in the fourth grade. It involves the story of Galileo being forced to recant his views that the Sun, rather than Earth, is the center of the solar system. Even a child can appreciate the fact that if the church felt it was necessary to force Galileo to recant, then the stakes in the issue must have been high. But, what

[1] A companion book [90] is being prepared that addresses issues other than those described here.

difference does it make if the Sun revolves about the Earth, or the Earth about the Sun? After all, whichever occurs, one forms the center of a circular motion for the other. Why should we care which is which?

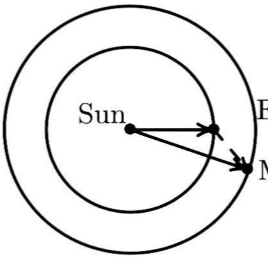

Fig. 1.1. Sun-Earth-Mars coordinates in half-astronomical units

1.1.1 Motion of Mars

To explain the kinds of difficulties that are introduced by an Earth-centered prejudice, start with the Sun as the center of our solar system. A simplified story has Mars approximately 3/2 times (actually, about 1.524 times) as far from the Sun as the Earth, and Mars takes approximately two years (about 687 Earth days) to complete its journey about the Sun.

To keep everything simple, eliminate fractions by replacing the standard astronomical unit (the distance between the Earth and the Sun) with what I call "half-astronomical" units. In the new system, which is depicted in Fig. 1.1, the Earth is two units from the Sun, and Mars is three. Using complex variables, a reasonable description of the motion of the Earth is given by $z_E(t) = 2e^{2\pi it}$ while that of Mars is $z_M = 3e^{\pi it}$.

Finding the orbit of Mars relative to the Earth now is simple; it is

$$z(t) = z_M(t) - z_E(t) = 3e^{\pi it} - 2e^{2\pi it}. \tag{1.1}$$

To describe this orbit, add and subtract the distance to the Sun to obtain

$$\begin{aligned} z(t) &= 3e^{\pi it} - 2e^{2\pi it} - 2 + 2 = 2 + e^{\pi it}[3 - 2e^{\pi it} - 2^{-\pi it}] \\ &= 2 + [3 - 4\cos(\pi t)]e^{\pi it}. \end{aligned} \tag{1.2}$$

According to Eq. 1.2, the graph of this equation, as given in Fig. 1.2, depicts the surprisingly complicated orbit of Mars when viewed relative to that of the Earth: it is *a limacon* with a nicely defined loop.[2]

[2]In my introductory calculus courses, I often use the trigonometric version of this

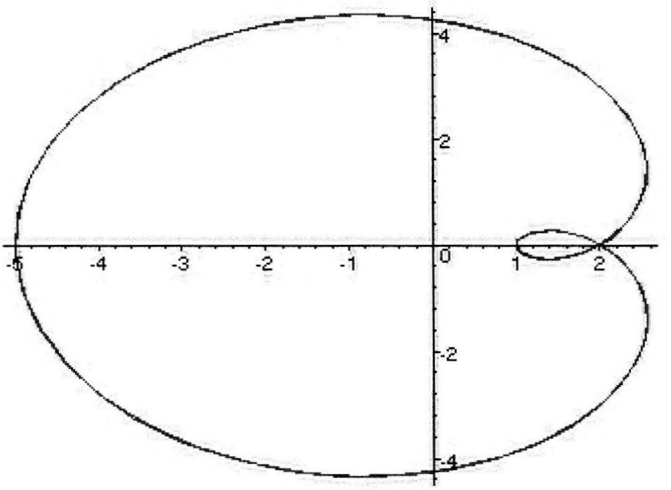

Fig. 1.2. Apparent motion of Mars relative to the Earth

Figure 1.2 makes it clear why the pre-Copernican, Earth-centered prejudice made it so difficult to predict the motion of the planets and to develop a "Newtonian Theory." For a segment of time on this orbit, everything is regular. Indeed, starting at the point where the loop intersects itself, Mars starts on its long journey moving away from the Earth until eventually it is five half-astronomical units away. (This position corresponds to where Earth and Mars are on opposite sides of the Sun.) The interesting, counterintuitive action starts when Mars returns to begin its close approach to the Earth. First, it quickly swoops in a radical plunge toward the Earth. But rather than colliding, Mars suddenly *reverses direction* to swoop out—a motion suggesting that the physics—for some strange reason—suddenly changes to a *law of repulsion* rather than attraction. Finally Mars changes direction once more so that it can repeat its long two-year journey.

Imagine the difficulty in determining the appropriate force law—a law that resembles some form of attraction for most of the journey only to suddenly become a law of repulsion when Mars approaches Earth, and then reverts back into a law of attraction. Other than resorting to bad jokes about the annoyance of Earthling's politics or their behavior, how does one explain the sudden repulsion of Mars when it starts approaching Earth? In other words, the change of variables from a Earth-centered to a Sun-centered system makes a considerable difference: without it, it is difficult to

example to put life into those mandatory reviews of trigonometry. The trigonometric version just uses double angle formulae; e.g., $(3\cos(\pi t), 3\sin(\pi t)) - 2(\cos(2\pi t), 2\sin(2\pi t)) = (2,0) + \rho(cos(\pi t), \sin(\pi t))$ where $\rho = 3 - 4\cos(\pi t)$.

even imagine how Newton's laws of attraction could have been developed.

Incidentally, it is easy to observe this retrograde behavior of Mars. Of course, the change in distance between Earth and Mars cannot be detected by the untrained naked eye, but the change in direction—where Mars appears to be moving in one direction, stops and moves backwards, and then stops again to return to its original direction—is quite apparent over the span of several nights. During those periods when Mars approaches Earth to start its dipping behavior, even a casual observer can notice how at a fixed time each night the position of Mars swings to define, over a period of days, a compressed "Z."

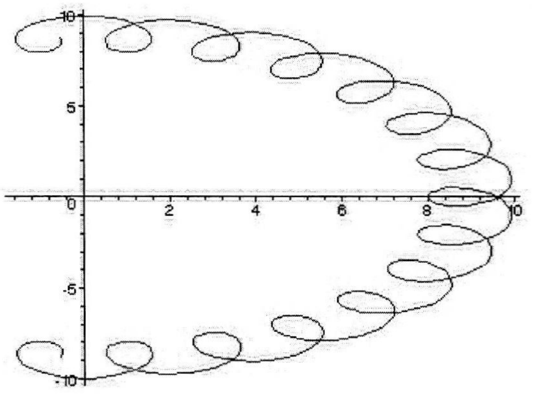

Fig. 1.3. Apparent orbit of a planet 9 times farther from the Sun

While the apparent motion of Mars offers surprising behavior, the orbits of the planets farther from the Sun adopt a much more complicated appearance with the several loops as indicated in Fig. 1.3. This figure depicts the apparent behavior of a planet nine AU away from the Sun: a distance that is a bit short of Saturn's actual orbit. Rather than developing a complicated version of the above description, a different elementary approach is described next.

1.1.2 The "far out" planets

Consider the circular orbit of a far-out planet—Mars, Saturn, or beyond—given by $z_P(t) = ae^{\alpha\pi it}$ where the value of $a \geq 3$ defines the distance from the Sun in our half-astronomical units: the α values are discussed below. After expressing this

$$z(t) = z_P(t) - z_E(t) = ae^{\alpha\pi it} - 2e^{2\pi it}, \tag{1.3}$$

orbit of the planet relative to the Earth in the usual complex variable form of $z(t) = r(t)e^{i\theta(t)}$, a way to determine whether the orbit is moving in a clockwise or counter-clockwise manner (relative to the Earth) is to examine the sign of $\theta'(t)$.

The sign of $\theta'(t)$ is the imaginary part of $(\ln z_P(t))' = \frac{z'_P}{z_P} = \frac{r'}{r} + i\theta'$. But since

$$(\ln z_P(t))' = \frac{z'_P}{z_P} = \frac{\pi i(a\alpha - 4e^{(2-\alpha)\pi i t})}{a - 2e^{(2-\alpha)\pi i t}}, \tag{1.4}$$

it follows from the form of the numerator that the sign of θ' must change periodically whenever $a\alpha < 4$.

The reason this $a\alpha < 4$ inequality must hold for all of the planets that are farther from the Sun than the Earth is *Kepler's third law.* This law asserts that

$$a^3\alpha^2 = k \tag{1.5}$$

where k is a constant. Consequently, $a\alpha = (\frac{k}{a})^{1/2}$ is a decreasing function of a: remember, a is the distance of the planet to the Sun. Thus, for a planet sufficiently far from the Sun, we must expect its orbit to experience loops when expressed relative to the Earth. According to Eq. 1.4, the loop occurs whenever the distance between the Earth and the planet decreases toward a (local) minimal value. But because those far-out planets take from decades to a couple of Earth centuries to circle the Sun,[3] it follows that their apparent orbits must exhibit many loops.

A natural related question, which is needed for later purposes, is to determine how far a planet must be beyond the Earth so that its apparent orbit has a loop. Using the units of the Earth, $a = 2, \alpha = 2$, we have that $k = 32$ for Eq. 1.5. Thus, $a^3\alpha^2 = 32$, or the crucial parameter has the value $a\alpha = [32/a]^{\frac{1}{2}}$. Because apparent loops occur when $a\alpha < 4$, it follows that these loops occur when $[32/a]^{\frac{1}{2}} < 4$, or when $a > 2$. Restated in words,

> *the apparent motion of any planet that is farther from the Sun than the Earth has a loop.*

Of course, this assertion holds for all bodies governed by Newton's equation: this fact plays a key role in the discussion about the rings of Saturn given in the last section of this chapter.

Notice how this simple argument just describes a circular uniform motion relative to another circular uniform orbit. The importance of this comment

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[3]While Venus takes only about 224 Earth days to circle the Sun, Jupiter takes 4332 (about 11.9 Earth years), Saturn 10,760 (about 29.5 years), Uranus 30,685 (about 84 years), Neptune 60,190 (about 165 years), and Pluto 90,800 days (about 249 years).