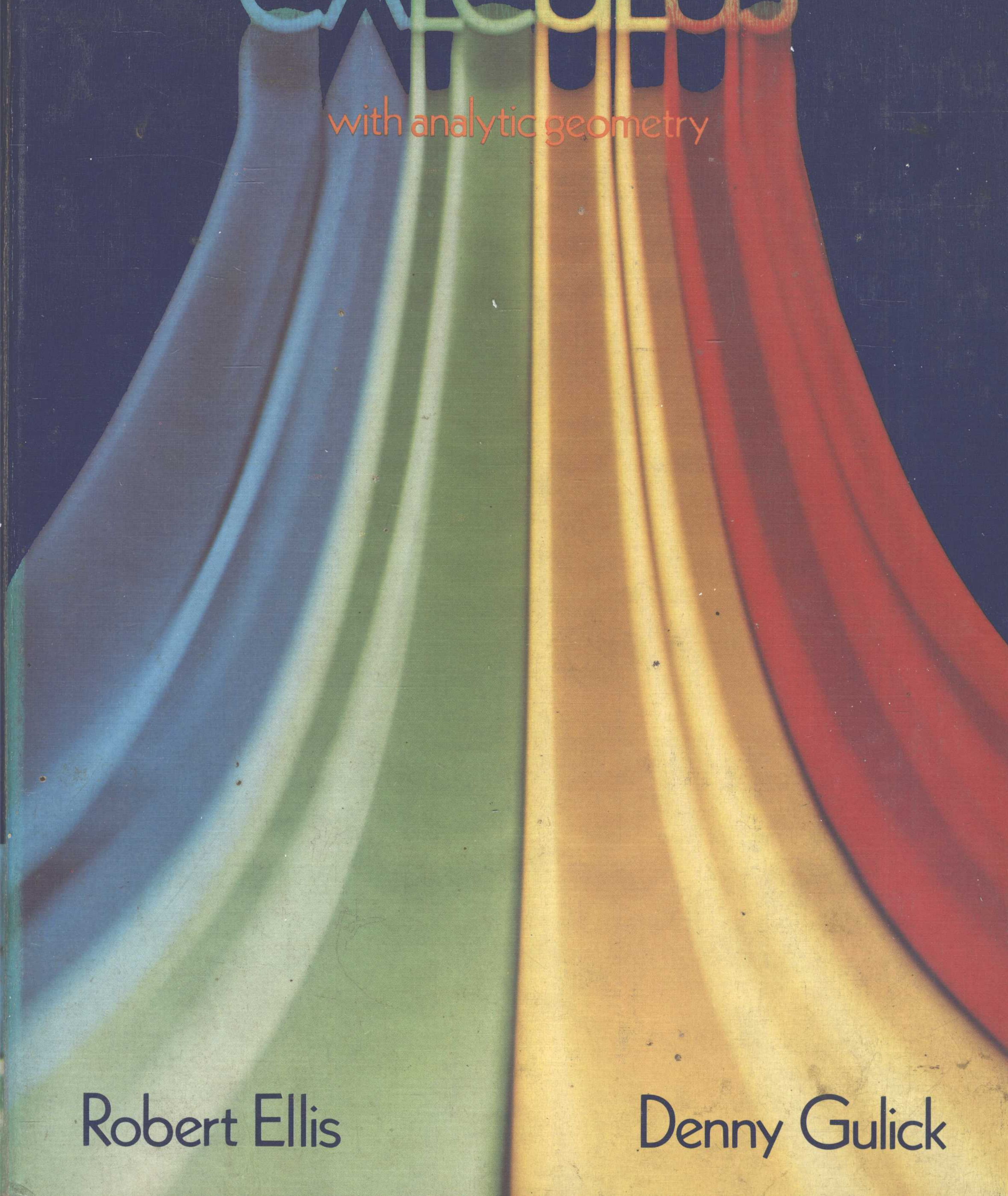
# CALCULUS

## with analytic geometry



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# Preface

This book contains all the topics that normally constitute a course in calculus of one and several variables. It is suitable for sequences taught in three semesters or in four or five quarters. In the three-semester case, the first semester will normally include the introductory chapter (Chapter 1), the three chapters on limits and derivatives (Chapters 2–4), and the initial chapter on integrals (Chapter 5). The second semester would then include the remaining chapters involving integration (Chapters 6–8) and some combination of the chapters on series (Chapter 9), conic sections (Chapter 10), and the introduction to vectors and vector-valued functions (Chapters 11 and 12). In the third semester the final chapters would be covered; these concern calculus of several variables and culminate in Green's Theorem, Stokes's Theorem, and the Divergence Theorem.

Although we develop the material in the order that we have found pedagogically most effective, instructors will have considerable flexibility in choosing topics. Chapter 1 (which includes a section on trigonometry, so that trigonometric functions can serve as examples throughout the book) is preliminary and can be covered quickly if the student's preparation is sufficient. We have made Chapter 9 virtually independent of the chapters that follow, so that it can be taken out of sequence; Chapter 10 can be studied any time after Chapter 4. Finally, we have included an appendix on differential equations (Appendix B) for optional use during the second or third semester.

Whenever possible, we use geometric and intuitive motivation to introduce concepts and results, so that students may readily absorb the carefully worded definitions and theorems that follow. The topical development, in which we employ numerous worked examples and some 780 illustrations, aims for clarity and precision without overburdening the reader with formalism. In keeping with this goal, we have proved most theorems of first-year calculus in the main body of the text but have placed the more difficult proofs in Appendix A. In the chapters on calculus of several variables we have proved selected theorems that aid comprehension of the material.

There are more than 5500 exercises, which we have placed both at the ends of sections and, for review, at the end of each chapter. Each set begins with a full complement of routine exercises to provide practice in using the ideas and methods presented in the text. These are followed by applied problems and by other exercises of a more challenging nature (indicated with an asterisk). To supplement the usual problems from physics and engineering, we have included many from business, economics, biology, chemistry, and other disciplines, as well as a small number of exercises suitable for solution on a calculator. In the interest of accuracy every exercise has been completely worked by each of the authors and by at least one other person. Answers to odd-numbered exercises and to word problems (except those requiring longer explanations or graphs) appear at the back of the book, and the accompanying Solutions Manual contains complete solutions to all exercises.

Throughout the book, statements of definitions, theorems, lemmas, and corollaries, as well as important formulas, are highlighted with tints for easy identification. Numbering is consecutive throughout each chapter for definitions and theorems, and consecutive within each section for examples and formulas. We use the symbol ■ to signal the end of a proof and □ for the end of the solution to an example.

Lists of Key Terms and Expressions, Key Formulas, and Key Theorems appear at the end of each chapter. On the endpapers we have assembled important formulas and results that the student will want to have handy, both for course review and for reference in later studies. Pronunciation of difficult terms and names is shown in footnotes on the pages where they first appear.

We are very grateful to many people who have helped us in a variety of ways as we prepared this book.

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# To the Reader

When you begin to study calculus, you will find that you have encountered many of its concepts and techniques before. Calculus makes extensive use of plane geometry and algebra, two branches of mathematics with which you are already familiar. However, added to these is a third ingredient, which may be new to you: the notion of limit and of limiting processes. From the idea of limit arise the two principal concepts that form the nucleus of calculus; these are the derivative and the integral.

The derivative can be thought of as a rate of change, and this interpretation has many applications. For example, we may use the derivative to find the velocity of an object, such as a rocket, or to determine the maximum and minimum values of a function. In fact, the derivative provides so much information about the behavior of functions that it greatly simplifies graphing them. Because of its broad applicability, the derivative is as important in such disciplines as physics, engineering, economics, and biology as it is in pure mathematics.

The definition of the integral is motivated by the familiar notion of area. Although the methods of plane geometry enable us to calculate the areas of polygons, they do not provide ways of finding the areas of plane regions whose boundaries are curves other than circles. By means of the integral we can find the areas of many such regions. We will also use it to calculate volumes, centers of gravity, lengths of curves, work, and hydrostatic force.

The derivative and the integral have found may diverse uses. The following list, taken from the examples and exercises in this book, illustrates the variety of the fields in which these powerful concepts are employed.

| | Section |
|---|---|
| Contraction of windpipe during coughing | 4.3 |
| Cost of insulating an attic floor | 4.4 |
| Force of water on an earth-filled dam | 7.6 |
| Amount of anesthetic needed during an operation | 8.5 |
| Age of a fossil, skull, or lunar rock | 8.5 |
| Orbit of a planet | 12.6 |
| Escape velocity of a rocket from the earth's gravitational field | 12.6 |
| Analysis of a rainbow | 13.3 |
| Effect of taxation on the production of a commodity | 13.3 |
| Electric field produced by a charged telephone wire | 15.5 |

The concepts basic to calculus can be traced, in uncrystallized form, to the time of the ancient Greeks. However, it was only in the sixteenth and early seventeenth centuries that mathematicians developed refined techniques for determining tangents to curves and areas of plane regions. These mathematicians and their ingenious techniques set the stage for Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), who are usually credited with the "invention" of calculus because they codified the techniques of calculus and put them into a general setting; moreover, they recognized the importance of the fact that finding derivatives and finding integrals are inverse processes.

During the next 150 years calculus matured bit by bit, and by the middle of the nineteenth century it had become, mathematically, much as we know it today. Thus the definitions and theorems presented in this book were all known a century ago. What is newer is the great diversity of applications, with which we will try to acquaint you throughout the book.

ROBERT ELLIS
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## 3 Derivatives



## 4 Applications of the Derivative



## 5 The Integral



## 6 Techniques of Integration

## 7 Applications of the Integral



## 8 Inverse Functions



## 9 Sequences and Series



## 10 Conic Sections

## 11 Vectors, Lines, and Planes



## 12 Vector-Valued Functions



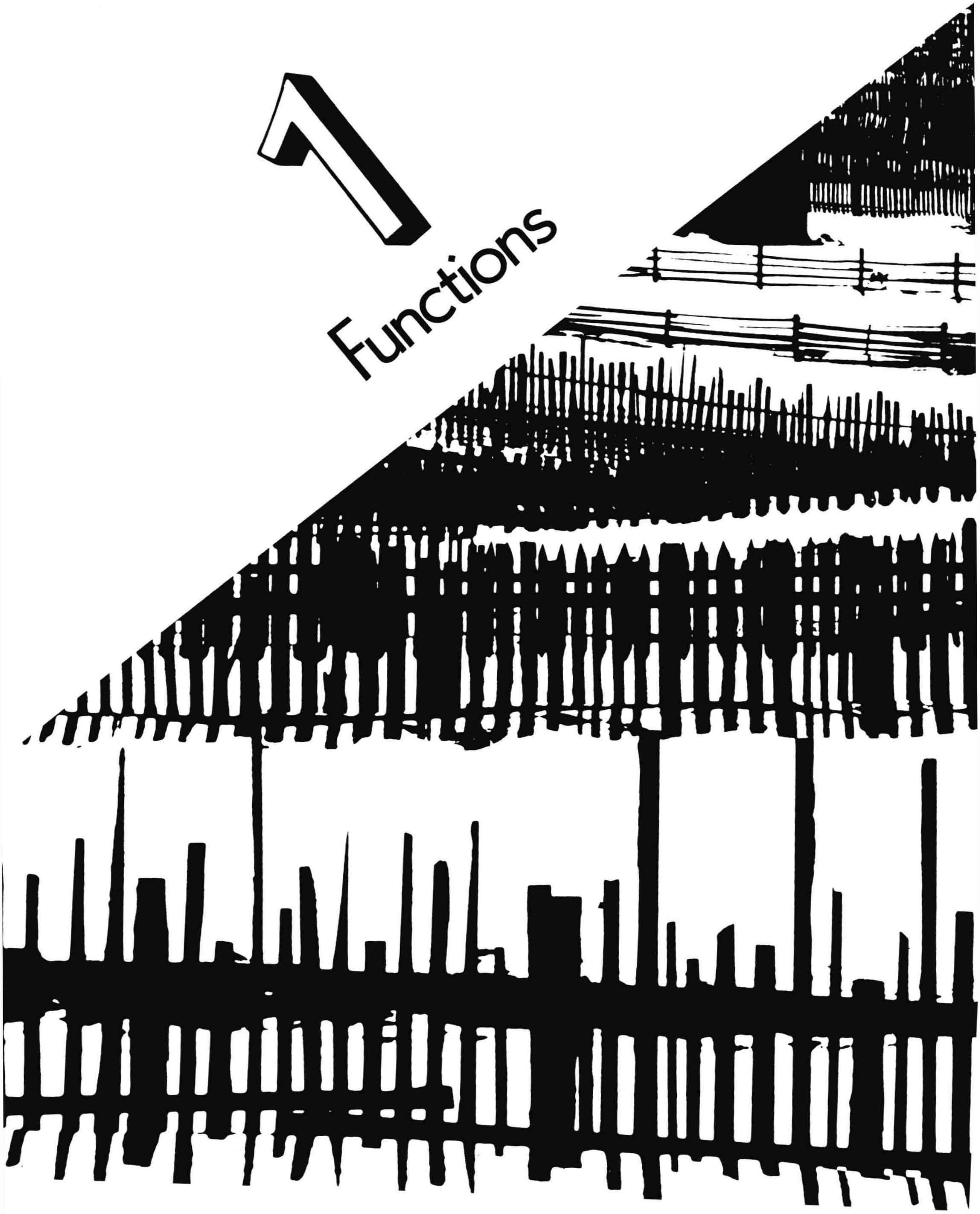
## 13 Partial Derivatives



## 14 Multiple Integrals

# 15 Calculus of Vector Fields

# 7 Functions



In this chapter we will review the basic properties of real numbers, introduce the concept of function, and discuss different types of functions. If you are already familiar with most of the definitions and concepts given, we suggest that you read Chapter 1 quickly and proceed to Chapter 2.

## 1.1 THE REAL NUMBERS

Real numbers, their properties, and their relationships are basic to calculus. Therefore we begin with a description of some important properties of real numbers. You should already be familiar with many of these properties.

### Types of Real Numbers and the Real Number Line

The best known real numbers are the *integers*:

$$0, \pm 1, \pm 2, \pm 3, \ldots$$

From the integers we derive the *rational numbers*. These are the real numbers that can be written in the form $p/q$, where $p$ and $q$ are integers and $q \neq 0$. Thus $\frac{48}{37}$, $-17$, and $-1.41$ are rational numbers. Any real number that is not rational is called an *irrational number*. Examples of irrational numbers are $\pi$ and $\sqrt{2}$. (See Exercise 58 at the end of this section for a proof that $\sqrt{2}$ is irrational.)

There is an order $<$ on the real numbers. If $a$ is not equal to $b$, then either $a < b$ or $a > b$. For example, $5 < 7$ and $-1 > -2$. If $a$ is less than or equal to $b$, we write $a \leq b$. If $a$ is greater than or equal to $b$, we write $a \geq b$. For example, $1 \leq x^2$ for any nonzero integer $x$. We say that $a$ is *positive* if $a > 0$ and *negative* if $a < 0$. If $a \geq 0$, we say that $a$ is *nonnegative*.

The real numbers can be represented as points on a horizontal line in such a way that if $a < b$, then the point on the line corresponding to the number $a$ lies to the left of the point on the line corresponding to the number $b$ (Figure 1.1). Such a line is called the *real number line*, or *real line*. We think of the real numbers as points on the real line, and *vice versa*. Thus we say that the negative numbers lie to the left of 0 and the positive numbers lie to the right of 0.

Sometimes in discussing collections of numbers, we use the symbol $\infty$ to denote "infinity" and $-\infty$ to denote "minus infinity." These symbols do

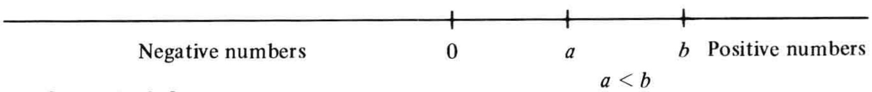



**FIGURE 1.1** The real line

not represent real numbers, but we will use them, as shown below, to describe certain intervals.

## Intervals

Certain sets of real numbers, called *intervals*, appear with great frequency in calculus. They can be grouped into nine categories:

| Name | Notation | Description |
|---|---|---|
| Open interval | $(a, b)$ | all $x$ such that $a < x < b$ |
| Closed interval | $[a, b]$ | all $x$ such that $a \leq x \leq b$ |
| Half-open interval | $(a, b]$ | all $x$ such that $a < x \leq b$ |
| Half-open interval | $[a, b)$ | all $x$ such that $a \leq x < b$ |
| Open interval | $(a, \infty)$ | all $x$ such that $a < x$ |
| Open interval | $(-\infty, a)$ | all $x$ such that $x < a$ |
| Closed interval | $[a, \infty)$ | all $x$ such that $a \leq x$ |
| Closed interval | $(-\infty, a]$ | all $x$ such that $x \leq a$ |
| The real line | $(-\infty, \infty)$ | all real numbers |

Intervals of the form $(a, b)$, $[a, b]$, $(a, b]$, and $[a, b)$ are *bounded intervals*, and intervals of the form $(a, \infty)$, $(-\infty, a)$, $[a, \infty)$, $(-\infty, a]$, and $(-\infty, \infty)$ are *unbounded intervals*. Figure 1.2 shows the four types of bounded intervals.

Note that $(b, b)$, $(b, b]$, and $[b, b)$ contain no numbers. More generally, $(a, b)$, $(a, b]$, and $[a, b)$ contain no numbers if $b \leq a$. Whenever we write one of these intervals, we make the implicit assumption that $a < b$. Likewise, we write $[a, b]$ only when $a \leq b$.

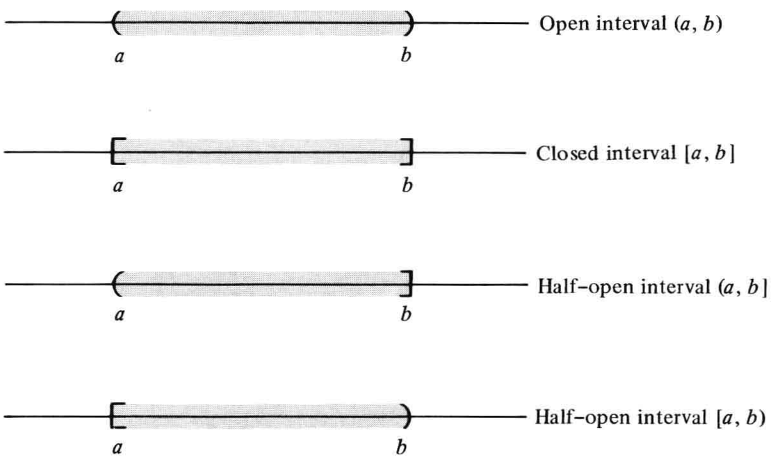



**FIGURE 1.2**

# Inequalities and Their Properties

Statements such as $a < b$, $a \leq b$, $a > b$, and $a \geq b$ are called *inequalities*. We list several basic laws for the inequalities $a < b$ and $a > b$. In what follows $a$, $b$, $c$, and $d$ are assumed to be real numbers.

Trichotomy: Either $a < b$, or $a > b$, or $a = b$, and only one of these holds for a given $a$ and $b$. (1)

Transitivity: If $a < b$ and $b < c$, then $a < c$. (2)

Additivity: If $a < b$ and $c < d$, then $a + c < b + d$. (3)

Positive multiplicativity: If $a < b$ and $c > 0$, then $ac < bc$. (4)

Negative multiplicativity: If $a < b$ and $c < 0$, then $ac > bc$. (5)

Replacing $<$ by $\leq$ and $>$ by $\geq$ in laws (2)–(5) yields four new laws for inequalities, which we will also find useful.

The word "trichotomy" in (1) means a threefold division. The trichotomy law states that any two numbers $a$ and $b$ are related in exactly one of the three ways listed in (1). For example, given the two numbers 3.1416 and $\pi$, we have either $3.1416 < \pi$, $3.1416 = \pi$, or $3.1416 > \pi$. (The last is actually correct.)

Sometimes, for simplicity of notation, two inequalities can be combined. For example, if $a \leq b$ and $b \leq c$, then we can write $a \leq b \leq c$.

*Caution:* The multiplication laws, (4) and (5), must be carefully observed. To illustrate their use we will solve several examples. In each, the problem is to "solve for $x$," which means to find all real numbers that satisfy a given inequality.

**Example 1** Solve the inequality $-x < -\frac{1}{3}$ for $x$.

**Solution** To find the values of $x$ that satisfy

$$-x < -\frac{1}{3}$$

we multiply each side of the inequality by $-1$, using rule (5). Thus

$$-x(-1) > -\frac{1}{3}(-1), \quad \text{or} \quad x > \frac{1}{3}$$

Therefore the solution is the open interval $(\frac{1}{3}, \infty)$. □

**Example 2** Solve $1/x < 3$ for $x$.

**Solution** The number $x = 0$ cannot be a solution, because division by 0 is impossible. We multiply through by $x$ to remove $x$ from the denominator. For $x$