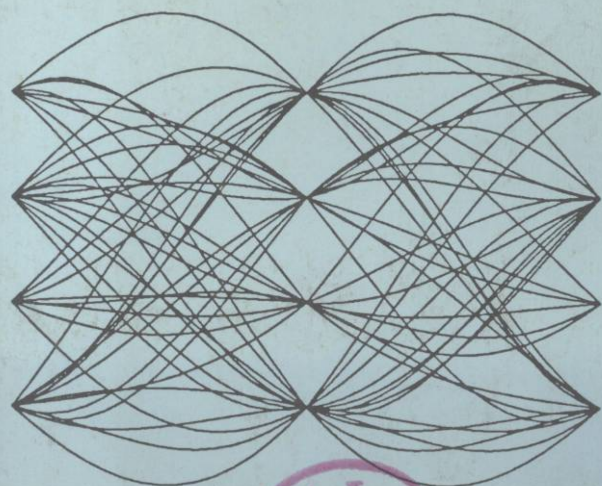
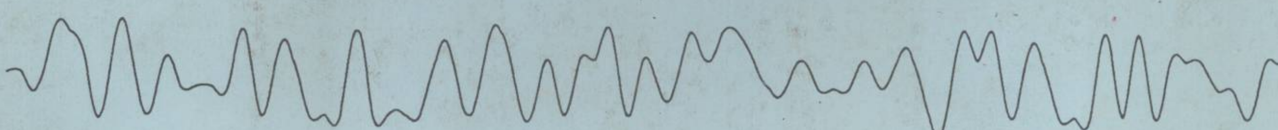
DIGITAL COMMUNICATION SOLUTIONS MANUAL



Edward A. Lee
David G. Messerschmitt



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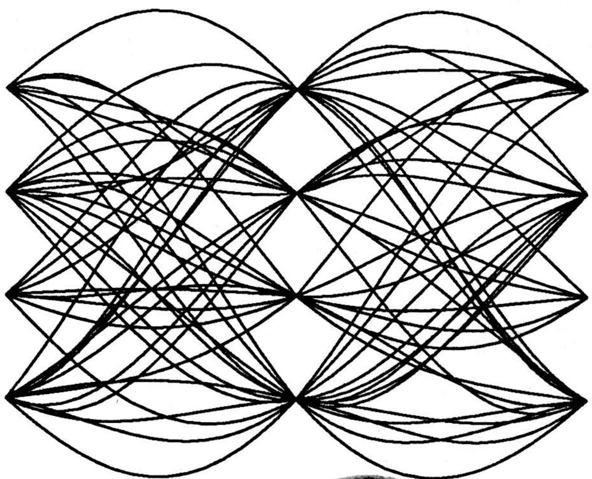

Edward A. Lee
David G. Messerschmitt

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CONTENTS

CHAPTER 1 SOLUTIONS

Problem 1-1.

a. The overload point of the A/D converter (largest signal that can be accommodated) will be chosen on the basis of the signal statistics and the signal power so as to keep the probability of overload low. Assuming the signal doesn't change, we would want to keep the overload fixed. Hence, the Δ would be halved.

b. Generally the error signal would be halved in amplitude. This would increase the SNR by $20\log_{10}2 = 6$ dB.

c. The bit rate would increase by f_s, the sampling rate.

d. We get, for some constant K,

$$SNR = 6n + K \,, \quad f_b = nf_s \,, \tag{1.1}$$

and thus

$$SNR = \frac{6f_b}{f_s} + K \,. \tag{1.2}$$

In particular, the SNR in dB is directly proportional to the bit rate.

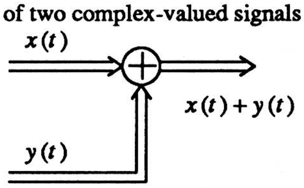
Problem 1-2. Each bit error will cause one recovered sample to be the wrong amplitude, which is similar to an added impulse to the signal. This will be perceived as a "pop" or a "click". The size of this impulse will depend on which of the n bits of a particular sample is in error. The error will range from the smallest quantization interval (the least-significant bit in error) to the entire range of signal levels (the sign bit in error).

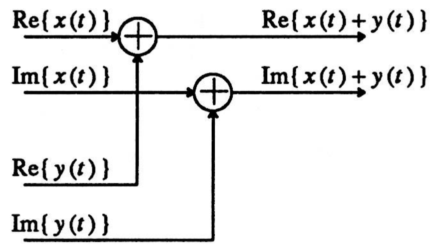
Problem 1-3. The most significant sources will be the anti-aliasing and reconstruction lowpass filters, which will have some group delay, and the propagation delay on the communication medium. Any multiplexes (chapter 16) will introduce a small amount of delay, as will digital switches (chapter 16).

Problem 1-4. Assume the constant bit rate is larger than the peak bit rate of the source. Then we might artificially increase the bit rate of the source up until it precisely equals the bit rate of the link by adding extra bits. We must have some way of identifying these extra bits at the receiver so that they can be removed. A number of schemes are possible, so here is but one: Divide the source bits in to groups called packets with arbitrary length. Append a unique sequence of eight bits, called a flag, to the beginning and end of each packet, and transmit these packets on the link interspersed with an idle code (say all zeros). The only problem now is to insure that the flag does not occur in the input bit stream. This can be accomplished using coding, with techniques described in chapter 16.

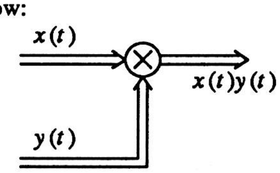
CHAPTER 2 SOLUTIONS

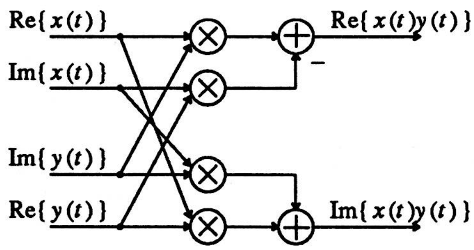
Exercise 2-1. Addition of two complex-valued signals is illustrated below:





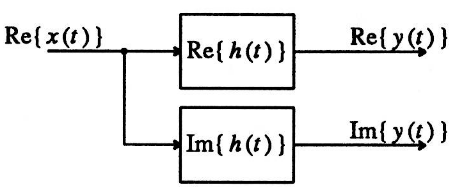
and multiplication below:

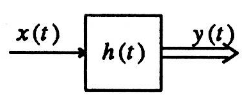




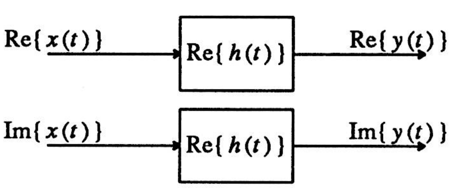
Complex addition is accomplished by two real additions, and complex multiplication by four real multiplications and two real additions.

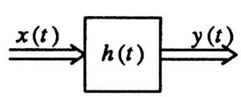
Exercise 2-2. A complex system with a real-valued input:





A real system with a complex-valued input:





Exercise 2-3. We can treat the convolution $x(t) * h(t)$ just like complex multiplication, since the convolution operation is linear — an integration. To check linearity, for a complex constant A and two input signals $x_1(t)$ and $x_2(t)$,

$$(x_1(t) + A \cdot x_2(t)) * h(t) = x_1(t) * h(t) + A \cdot (x_2(t) * h(t)) \tag{2.65}$$

following the rules of complex arithmetic. This establishes linearity.

Exercise 2-4.

$$Y(j\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau e^{-j\omega t}dt \,. \tag{2.66}$$

Observe that

$$e^{-j\omega t} = e^{-j\omega\tau}e^{-j\omega(t-\tau)} \tag{2.67}$$

so that

$$\begin{aligned} Y(j\omega) &= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \int_{-\infty}^{\infty} x(t-\tau)e^{-j\omega(t-\tau)}dt \\ &= H(j\omega)X(j\omega) \end{aligned} \tag{2.68}$$

after a change of variables.

Exercise 2-5. Take the Fourier transform of both sides of (2.2), getting

$$\begin{aligned} \hat{X}(j\omega) &= \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_m \delta(t-mT)e^{-j\omega t}dt \\ &= \sum_{m=-\infty}^{\infty} x_m \int_{-\infty}^{\infty} \delta(t-mT)e^{-j\omega t}dt \\ &= \sum_{m=-\infty}^{\infty} x_m e^{-j\omega mT} = X(e^{j\omega T}). \end{aligned} \tag{2.69}$$

Exercise 2-6. The impulse response of the system is

$$f_k = f(kT) \tag{2.70}$$

and hence (2.15) gives the transfer function directly,

$$F(e^{j\omega T}) = \frac{1}{T}\sum_m F[j(\omega + m\frac{2\pi}{T})] \,. \tag{2.71}$$

Exercise 2-7. Given $X(j\omega) = 0$ for all $|\omega| > \pi/T$, (2.15) implies that

$$X(e^{j\omega T}) = \frac{1}{T}X(j\omega) \quad \text{for all } |\omega| < \frac{\pi}{T} \,. \tag{2.72}$$

To get $x(t)$ from x_k, therefore, we can use

$$F(j\omega) = \begin{cases} T; & |\omega| < \pi/T \\ 0; & \text{otherwise} \end{cases} \tag{2.73}$$

in figure 2-1.

Exercise 2-8. First show that $S < \infty$ implies BIBO.

$$|y_k| = |\sum_{m=-\infty}^{\infty} h_m x_{k-m}| \le L \sum_{m=-\infty}^{\infty} |h_m| = LS < \infty. \tag{2.74}$$

Then show that if $S = \infty$ there exists a bounded input such that the output is unbounded. Such an input is

$$x_k = \begin{cases} h_{-k}^*/|h_{-k}|; & k \text{ such that } h_k \neq 0 \\ 0; & k \text{ such that } h_k = 0 \end{cases} \,. \tag{2.75}$$

Exercise 2-9. For all complex z such that $|z| \le 1$

$$|\sum_{k=-\infty}^{\infty} h_k z^{-k}| = |\sum_{k=-\infty}^{0} h_k z^{-k}| \le \sum_{k=-\infty}^{0} |h_k z^{-k}| \le \sum_{k=-\infty}^{\infty} |h_k| < \infty. \tag{2.76}$$

Exercise 2-10. The zero vector is

$$\mathbf{0} <\!\!-\!\!> (\cdots 0, \cdots, 0 \ldots) \tag{2.77}$$

or

$$\mathbf{0} <\!\!-\!\!> y(t) = 0\,. \tag{2.78}$$

The rest is a tedious but straightforward verification of the properties.

Exercise 2-11. This is a straightforward evaluation. For example,

$$<\mathbf{Y},\mathbf{X}> = \int_{-\infty}^{\infty} y(t) x^*(t)\, dt = (\int_{-\infty}^{\infty} x(t) y^*(t)\, dt)^* = <\mathbf{X},\mathbf{Y}>^*\,. \tag{2.79}$$

Exercise 2-12. Let $\mathbf{Y} \varepsilon M$, then

$$\begin{aligned} \|\mathbf{X}-\mathbf{Y}\|^2 &= \|\mathbf{X} - \mathbf{P}_M(\mathbf{X}) + \mathbf{P}_M(\mathbf{X}) - \mathbf{Y}\|^2 \\ &= \|\mathbf{X} - \mathbf{P}_M(\mathbf{X})\|^2 + \|\mathbf{P}_M(\mathbf{X}) - \mathbf{Y}\|^2 + 2<\mathbf{X}-\mathbf{P}_M(\mathbf{X}), \mathbf{P}_M(\mathbf{X}) - \mathbf{Y}> \end{aligned} \tag{2.80}$$

Since $(\mathbf{P}_M(\mathbf{X}) - \mathbf{Y}) \varepsilon M$ and $(\mathbf{X}-\mathbf{P}_M(\mathbf{X}))$ is orthogonal to the subspace M, the last term is 0 and

$$\begin{aligned} \|\mathbf{X}-\mathbf{Y}\|^2 &= \|\mathbf{X}-\mathbf{P}_M(\mathbf{X})\|^2 + \|\mathbf{P}_M(\mathbf{X}) - \mathbf{Y}\|^2 \\ &\ge \|\mathbf{X} - \mathbf{P}_M(\mathbf{X})\|^2 \end{aligned} \tag{2.81}$$

with equality if and only if $\mathbf{Y} = \mathbf{P}_M(\mathbf{X})$.

Exercise 2-13. The inequality is obviously true (with equality) if $\mathbf{X} = \mathbf{0}$ or $\mathbf{Y} = \mathbf{0}$, so assume that $\mathbf{X} \ne \mathbf{0}$ and $\mathbf{Y} \ne \mathbf{0}$. Then we have the inequality

$$\begin{aligned} 0 &\le \|\mathbf{X} - \alpha \cdot \mathbf{Y}\|^2 \\ 0 &\le \|\mathbf{X}\|^2 - 2\mathrm{Re}\{ \alpha^* <\mathbf{X},\mathbf{Y}> \} + |\alpha|^2 \|\mathbf{Y}\|^2 \end{aligned} \tag{2.83}$$

If we let

$$\alpha = \frac{<\mathbf{X},\mathbf{Y}>}{\|\mathbf{Y}\|^2} \tag{2.85}$$

then the previous inequality becomes

$$0 \le \|\mathbf{X}\|^2 - \frac{|<\mathbf{X},\mathbf{Y}>|^2}{\|\mathbf{Y}\|^2} \tag{2.86}$$

from which the Schwartz inequality follows immediately.

Problem 2-1. We start out with an easy problem! Looking at figure 2-2, when the imaginary part of the impulse response is zero, we see that the system consists of two independent filters, one for real part and one for imaginary part of the input, with no crosstalk. The imaginary part of the impulse response results in crosstalk between the real and imaginary parts.

Problem 2-2. Doing the discrete-time part only, write the convolution sum when the input is $e^{j\omega kT}$

$$y_k = \sum_{m=-\infty}^{\infty} e^{j\omega mT} h_{k-m}\,. \tag{2.87}$$

Changing variables,

$$\begin{aligned} y_k &= \sum_{n=-\infty}^{\infty} e^{j\omega(k-n)T} h_n \\ &= e^{j\omega kT} \sum_{n=-\infty}^{\infty} e^{-j\omega nT} h_n\,. \end{aligned} \tag{2.88}$$

The output is the same complex exponential multiplied by a sum that is a function of the impulse response of the system h_n and the frequency ω of the input, but is not a function of the time index k. This *frequency response* or *transfer function*

$$H(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} e^{-j\omega nT} h_n \tag{2.89}$$

is recognized as the *Fourier transform* of the discrete-time signal h_n.

Problem 2-3. The output of the impulse generator is defined as

$$\hat{w}(t) = \sum_{k=-\infty}^{\infty} w(k)\delta(t - kT) . \tag{2.90}$$

a.

$$\begin{aligned} Y(j\omega) &= F(j\omega) \sum_{k=-\infty}^{\infty} w_k e^{-j\omega kT} \\ &= F(j\omega)H(e^{j\omega T}) \cdot \frac{1}{T} \sum_{m=-\infty}^{\infty} G(j(\omega + m\frac{2\pi}{T}))X(j(\omega + m\frac{2\pi}{T})) \end{aligned} \tag{2.91}$$

b. Yes, you can see from a. that if we add two input signals, the output will be a similar superposition.

c. If $F(j\omega) = 0$ for $|\omega| > \pi/T$ then for $|\omega| \le \pi/T$ we have

$$Y(j\omega) = \frac{1}{T}F(j\omega)H(e^{j\omega T})G(j\omega)X(j\omega) \tag{2.92}$$

and the system is time-invariant with transfer function $\frac{1}{T}F(j\omega)H(e^{j\omega T})G(j\omega)$.

Problem 2-4. In continuous-time:

$$\begin{aligned} \int_{-\infty}^{\infty} |x(t)|^2 dt &= F.T.\left[x(t)x^*(t)\right]_{\omega=0} \\ &= \left[\frac{1}{2\pi}X(j\omega) * X^*(-j\omega)\right]_{\omega=0} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega . \end{aligned} \tag{2.93}$$

Discrete-time follows similarly.

Problem 2-5. We get that the energy of the discrete-time signal is

$$\sum_k |x_k|^2 = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} |\sum_m X(j(\omega + m\frac{2\pi}{T}))|^2 d\omega \tag{2.94}$$

and there is evidently no way to relate this to the energy of the continuous-time signal. However, if the continuous-time signal is properly bandlimited, then the sum inside the integral includes one term, and the right hand side is proportional to the energy. In fact, the energy of the discrete-time signal in this case is $1/T^2$ times the energy of the continuous-time signal. As the sampling rate increases, the energy of the discrete-time signal grows without bound, since we have more and more samples in the summation.

Problem 2-6. The transfer function is $H(z) = 1 + z^{-1}$ or $H(e^{j\omega T}) = 1 + e^{-j\omega T}$. The output is $y_k = A\cos(\omega_0 kT + \theta)$ where the magnitude response is $A = \sqrt{2(1 + \cos(\omega_0 T))}$ and the phase response is

$$\theta = \tan^{-1}\left[\frac{\sin(\omega_0 T)}{1 + \cos(\omega_0 T)}\right] = \tan^{-1}\left[\frac{2\sin(\frac{\omega_0 T}{2})\cos(\frac{\omega_0 T}{2})}{2\cos^2(\frac{\omega_0 T}{2})}\right] = \frac{\omega_0 T}{2} . \tag{2.95}$$

The phase is linear in ω_0.

Problem 2-7. The Fourier transform of a real system is conjugate symmetric, so

$$H(j\omega) = A(\omega)e^{j\theta(\omega)} = H^*(-j\omega) = A(\omega)e^{-j\theta(-\omega)}, \tag{2.96}$$

since $A(\omega) = |H(j\omega)|$ is both non-negative and symmetric. Hence, $\theta(\omega) = -\theta(-\omega)$.

Problem 2-8. From problem 2-7 the phase response of a real system is anti-symmetric, so the transfer function of the phase shifter should be

$$H(j\omega) = e^{-j\theta \mathrm{sgn}(\omega)} = \cos(\theta) + j\,\mathrm{sgn}(\omega)\sin(\theta), \tag{2.97}$$

where we have used the symmetry and anti-symmetry of the cos and sin, respectively. This becomes

$$h(t) = \delta(t)\cos(\theta) - \frac{1}{\pi t}\sin(\theta). \tag{2.98}$$

Problem 2-9. If $\mathrm{Re}\{\alpha\} > 0$, then we can use the Fourier transform pair

$$y(t) = \frac{1}{jt + \alpha} \;<\!->\; Y(j\omega) = 2\pi e^{\alpha\omega} u(-\omega) \tag{2.99}$$

where $u(\omega)$ is the unit step function. Then we observe that $x(t)$ is $y(t)$ convolved with an impulse stream $\sum_{m=-\infty}^{\infty} \delta(t - mT)$, so its transform is

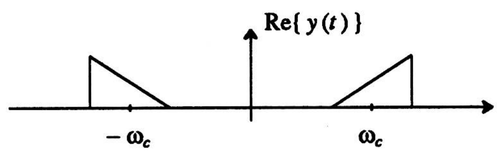
$$\begin{aligned} X(j\omega) &= Y(j\omega)\frac{2\pi}{T}\sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}m) \\ &\frac{2\pi^2}{T}\sum_{m=-\infty}^{0} \delta(\omega - \frac{2\pi}{T}m)e^{2\pi m\alpha/T}. \end{aligned} \tag{2.100}$$

If $\mathrm{Re}\{\alpha\} = 0$, then we can use the transform of $1/jt$, convolving it again with an impulse stream to get

$$\frac{-2\pi^2}{T}\sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}m)\mathrm{sgn}(\omega). \tag{2.101}$$

Problem 2-10. Given $Y(j\omega) = H(j\omega)X(j\omega)$, then if $X(j\omega_0) = 0$ then $Y(j\omega_0) = 0$.

Problem 2-11.



Yes, this is bandwidth efficient.

Problem 2-12. From (2.6),

$$y_k = \sum_{m=-\infty}^{\infty} z^m h_{k-m}. \tag{2.102}$$

Changing variables,

$$y_k = \sum_{n=-\infty}^{\infty} z^{k-n} h_n = z^k \sum_{n=-\infty}^{\infty} z^{-n} h_n = z^k H(z). \tag{2.103}$$

Because the system is time invariant, $H(z)$ does not depend on k.

Problem 2-13. Let the response to z^k be y_k. By linearity and problem 2-1, if the input to the system is z^{k+m} then the output is $z^m y_k = z^k z^m$. By time invariance the response to that same input is y_{k+m}. Setting these two responses equal,

$$z^m y_k = y_{k+m} \tag{2.104}$$

and setting $k = 0$ we get the desired result

$$y_m = y_0 z^m. \tag{2.105}$$

The transfer function is complex number y_0, which is evidently a function of z, so we define the notation $y_0 = H(z)$ to reflect this property.

Problem 2-14. The Z transform is

$$\begin{aligned} X(z) &= \sum_{m=-\infty}^{\infty} z^{-m} a^m u_m \\ &= \sum_{m=0}^{\infty} (az^{-1})^m. \end{aligned} \tag{2.106}$$

For any b such that $|b| < 1$ we have the identity

$$\frac{1}{1-b} = \sum_{m=0}^{\infty} b^m \tag{2.108}$$

which is easily verified by using long division on the left hand side. Therefore, in the region $|z^{-1}| < \frac{1}{|a|}$, the Z transform is

$$X(z) = \frac{1}{1-az^{-1}}. \tag{2.109}$$

Outside of this region, the Z transform does not exist. If $|a| > 1$, we easily see from (2.57) that the signal goes to infinity as k increases. Not coincidentally, the region in which the Z transform exists does not include the unit circle, implying that the Fourier transform does not exist either.

Problem 2-15. a. If the response of the system to z^t is $y(t)$, then by linearity the response to $z^{t+u} = z^t z^u$ is $z^u y(t)$. By time invariance, the response to z^{t+u} is $y(t+u)$. Setting these two equal, $y(t+u) = t^u y(t)$, and setting $t = 0$, $y(u) = y(0)z^u$.

b. Clearly $e^{st} = z^t$ if $z = e^s$, i.e., if $s = j\omega$, then $z = e^{j\omega}$, a point on the unit circle.

c. Substituting into the convolution,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau = e^{st}H(s) \tag{2.110}$$

which implies that $H(s)$ is an eigenvalue of the system.

Problem 2-16. The solution is basically the same as for problem 2-13.

Problem 2-17.

a.

$$X(z) = \frac{z}{z-a} \tag{2.111}$$

in both cases.

b. $|z| > |a|$ and $|z| < |a|$ respectively.

c. $|a| < 1$ and $|a| > 1$ respectively.

Problem 2-18. Since

$$\frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots \tag{2.112}$$

we get the signal of problem 2-17a, where the region of convergence is $|az^{-1}| < 1$ or $|z| > |a|$. Also, since

$$-\frac{\frac{z}{a}}{1-\frac{z}{a}} = -\frac{z}{a}\left(1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \cdots\right) \tag{2.113}$$

we get the signal of problem 2-17b where the region of convergence is $|za^{-1}| < 1$ or $|z| < a$.

Problem 2-19. First we perform a partial fraction expansion,

$$X(z) = \frac{A}{z-a} + \frac{B}{z-b} \tag{2.114}$$

where

$$A = \frac{a^2}{a-b},\quad B = \frac{b^2}{b-a}. \tag{2.115}$$

a. The region of convergence is $|z| > |b|$, and applying problem 2-17a to both terms in the partial fraction expansion,

$$x_k = A\cdot a^k + B\cdot b^k \tag{2.116}$$

for $k \ge 0$, and zero otherwise.

b. The region of convergence must be $|a| < |z| < |b|$ and hence, applying the results of problem 2-17b,

$$x_k = \begin{cases} A\cdot a^k, & k \ge 0 \\ -B\cdot b^k, & k < 0 \end{cases} \tag{2.117}$$

c. For (a) the signal is not stable because $b^k \to \infty$. This is because the region of convergence does not include the unit circle. For (b) the region of convergence does include the unit circle so the signal is stable (this is the only region of convergence for which the signal is stable).

Problem 2-20. $H^*(j\omega)$.

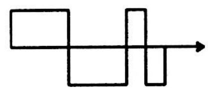
Problem 2-21. a. The norm of both signals is unity. The inner product is

$$<S_1,S_2> = \int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt = 1\cdot\frac{3}{4} - 1\cdot\frac{1}{4} = \frac{1}{2}\,. \tag{2.118}$$

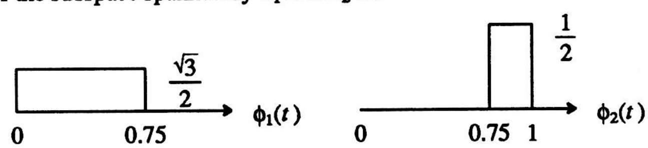
b.

$$\| S_1 + S_2 \|^2 = \int_{-\infty}^{\infty} (s_1(t) + s_2(t))^2 dt = 4\cdot\frac{3}{4} + 0\cdot\frac{1}{4} = 3\,. \tag{2.119}$$

c. There are many possibilities, but here is one:



d. Define an orthonormal basis for the subspace spanned by S_1 and S_2 as:



A signal orthogonal to S_1 that is a linear combination of Φ_1 and Φ_2 is $2\cdot\Phi_1 - \frac{2}{\sqrt{3}}\cdot\Phi_2$.

e. The projection of S_5 on the two basis vectors is

$$<S_5,\Phi_1> = -\frac{\sqrt{3}}{8} \quad <S_5,\Phi_2> = -\frac{1}{8} \tag{2.120}$$

and hence the projection on the subspace is $-\frac{\sqrt{3}}{8}\cdot\Phi_1 - \frac{1}{8}\cdot\Phi_2$.

Problem 2-22. a. Clearly if two signals are bandlimited, then their weighted sum is also bandlimited.

b. Let X be in the subspace. By Parseval's theorem (problem 2-4), for any $Y \varepsilon B$,

$$\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \frac{1}{2\pi}\int_{-W}^{W} X(j\omega)Y^*(j\omega)\,d\omega = 0\,. \tag{2.121}$$

Clearly, this is satisfied if and only if $X(j\omega) = 0$ for $|\omega| \le W$.

c. Let this projection be **P**, then $<S_1 - P,Y> = 0$ for all $Y \varepsilon B$. From b. this implies that $S_1(j\omega) = P(j\omega)$ for $|\omega| \le W$, and of course since $P \varepsilon B$, we must have that $P(j\omega) = 0$ for $|\omega| > W$. Hence,

$$S_1(j\omega) = \int_0^1 e^{-j\omega t}\,dt = \frac{1 - e^{-j\omega}}{j\omega} \tag{2.122}$$

and

$$p(t) = \frac{1}{2\pi}\int_{-1}^{1} \frac{1 - e^{-j\omega}}{j\omega} e^{j\omega t}\,d\omega\,. \tag{2.123}$$

Unfortunately this integral cannot be evaluated in closed form.

Problem 2-23. Let $X_1 \varepsilon M_1$ and $X_2 \varepsilon M_2$. An element of $M_1 \oplus M_2$ can be written in the form $X_1 + X_2$. Hence it suffices to show that

$$<X-(P_{M_1}(X) + P_{M_2}(X)),X_1 + X_2> = 0 \tag{2.124}$$

Expanding the left side, it equals

$$<X-P_{M_1}(X),X_1> - <P_{M_2}(X),X_1> + <X - P_{M_2}(X),X_2> - <P_{M_1}(X),X_2> \tag{2.125}$$

But by the definition of projection

$$<X - P_{M_1}(X),X_1> = <X - P_{M_2}(X),X_2> = 0 \tag{2.126}$$

and because M_1 and M_2 are orthogonal subspaces

$$<\mathbf{P}_{M_2}(\mathbf{X}),\mathbf{X}_1> = <\mathbf{P}_{M_1}(\mathbf{X}),\mathbf{X}_2> = 0 \tag{2.127}$$

and hence the result is established.

Problem 2-24. Defining

$$z^{-k}\cdot\mathbf{H} <-> h(t - kT)\ , \tag{2.128}$$

the Schwarz inequality states that

$$|\rho_h(k)| \leq ||\mathbf{H}||\cdot||z^{-k}\cdot\mathbf{H}|| \tag{2.129}$$

and since it is easy to verify that the signal and its time-translate have identical norms, this becomes the desired result.

CHAPTER 3 SOLUTIONS

Exercise 3-1. This follows from

$$E[e^{s(X+Y)}] = E[e^{sX}]E[e^{sY}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{sx}f_X(x)e^{sy}f_Y(y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty} e^{sx}f_X(x)\,dx \int_{-\infty}^{\infty} e^{sy}f_Y(y)\,dy = \Phi_X(x)\Phi_Y(y)\ . \tag{3.186}$$

Exercise 3-2. Evaluating the derivatives,

$$\Phi_X(s)\,\Big|_{s=0} = 1\ , \tag{3.187}$$

$$\frac{\partial}{\partial s}\Phi_X(s)\,\Big|_{s=0} = E[Xe^{sX}]\,\Big|_{s=0} = E[X]\ , \tag{3.188}$$

$$\frac{\partial^2}{\partial s^2}\Phi_X(s)\,\Big|_{s=0} = E[X^2e^{sX}]\,\Big|_{s=0} = E[X^2]\ . \tag{3.189}$$

Exercise 3-3.

a. The distribution function can be written in terms of a unit step function $u(x)$ as

$$1 - F_X(x) = \int_{-\infty}^{\infty} u(y-x)\,dF_X(y) = \int_{x}^{\infty} dF(y) \tag{3.190}$$

and since $u(y-x)$ is bounded by $e^{(y-x)s}$ for $s \geq 0$,

$$1 - F_X(x) \leq \int_{-\infty}^{\infty} e^{(y-x)s}\,dF_X(y) = e^{-sx}\Phi_X(s). \tag{3.191}$$

b. Obtained by a similar technique.

c. Take the derivative of the bound w.r.t. s and set to zero.

Exercise 3-4. Suppose that y is a discrete value that Y takes on with probability a. Then

$$f_Y(\alpha) = a\,\delta(\alpha - y). \tag{3.192}$$

Integrate (3.30) over small intervals about y, or over $(y-\varepsilon, y+\varepsilon)$ for small enough ε. Equation (3.32) follows similarly, or it can be easily derived from the definition of conditional probabilities (3.27).

Exercise 3-5. By direct calculation we have

$$\Pr[X > x] = \frac{1}{\sigma\sqrt{2\pi}}\int_{x}^{\infty} e^{-(\alpha-\mu)^2/2\sigma^2}\,d\alpha = \frac{1}{\sigma\sqrt{2\pi}}\int_{(x-\mu)/\sigma}^{\infty} e^{-w^2/2}\sigma\,dw = Q\left(\frac{x-\mu}{\sigma}\right), \tag{3.193}$$

where we have used the change of variables $w = (\alpha - \mu)/\sigma$.

Exercise 3-6.

a. The moment generating function can be obtained by evaluating the integral

$$\Phi_X(s) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx} e^{-(x-\mu)^2/2\sigma^2} dx \ . \tag{3.194}$$

Completing the square in the exponent in the integral, it becomes $-((x-a)^2 - b)/2\sigma^2$ where $b = 2\mu\sigma^2 s + \sigma^4 s^2$. The $e^{b/2\sigma^2}$ term can be taken outside the integral, and the remaining integrand is just a Gaussian density function and therefore integrates to unity. Thus, the moment generating function is $\Phi_X(s) = e^{b/2\sigma^2}$, and substituting for b we get the claimed result.

b. The optimal value of s can be solved as

$$s = \frac{x-\mu}{\sigma^2} \tag{3.195}$$

and hence the bound is valid as long as $x \geq \mu$. Substituting this value of s into the bound, we get

$$1 - F_X(x) \leq e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{3.196}$$

which looks remarkably like the Gaussian density. Note than when $x = \mu$, the actual probability is ½ and the Chernoff bound is $e = 2.28$, so the bound is rather loose. It becomes much tighter for larger values of x. The relation (3.43) follows by letting $\mu = 0$ and $\sigma = 1$.

Exercise 3-7. Consider a scaled Gaussian, $Y = aX$. If the variance of X is σ^2, then the variance of Y is $a^2\sigma^2$. Hence the moment generating function of Y is

$$\Phi_Y(s) = e^{a^2\sigma^2 s^2/2}. \tag{3.197}$$

The moment generating function is

$$\Phi_Z(s) = e^{(a_1^2 + \cdots + a_N^2)\sigma^2 s^2/2}. \tag{3.198}$$

This is the moment generating function of a zero mean Gaussian random variable with variance (3.46).

Exercise 3-8. We only need to show

$$\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y] \Rightarrow f_{X,Y}(x,y) = f_X(x) f_Y(y). \tag{3.199}$$

From (3.48) and the fact that the random variables have zero-mean, $\rho = 0$. Now (3.47) is easily factored into two parts.

Exercise 3-9. First show that

$$R_W(\tau) = \mathrm{E}[W(t+\tau)W^*(t)] = h^*(-\tau) * R_{WX}(\tau) \tag{3.200}$$

by substituting for one of the $W(t)$ in terms of $X(t)$. Then show that

$$R_{WX}(\tau) = h(\tau) * R_X(\tau), \tag{3.201}$$

completing the result. Finally, show that the Fourier transform of $h^*(-\tau)$ is $H^*(j\omega)$.

Exercise 3-10. Calculating first the cross-correlation of input and output,

$$R_{XW}(m) = E[X_{k+m} W_k^*] = E[X_{k+m} \sum_{n=-\infty}^{\infty} X_n^* h_{k-n}^*] = R_x(m) * h^*_{-m} \ . \tag{3.202}$$

Then calculating the output correlation function,

$$R_W(m) = E[W_{k+m} W_k^*] = E[\sum_{n=-\infty}^{\infty} X_n h_{k+m-n} W_k^*] = R_{XW}(m) * h_m \ . \tag{3.203}$$

Finally, show that the Fourier transform of h^*_{-m} is $H^*(e^{j\omega T})$.

Exercise 3-11. First we can calculate that

$$R_{WY}(\tau) = R_{XY}(\tau) * h(\tau) \tag{3.204}$$

and then

$$R_{WU}(\tau) = R_{WY}(\tau) * g^*(-\tau) = R_{XY}(\tau) * h(\tau) * g^*(-\tau) \ . \tag{3.205}$$

Taking the Fourier transform we get the desired result.

Exercise 3-12. This relation can be obtained by exactly the same method as Appendix 3-A, although it is tedious.

Exercise 3-13. We use the fact that the next state Ψ_{k+1} of a Markov chain is independent of the past states $\Psi_{k-1}, \Psi_{k-2}, \cdots$ given the present state Ψ_k to show that *all* future samples of the Markov chain are independent of the past given knowledge of the present.

We wish to show that for any $n > 0$ and any k,

$$p(\Psi_{k+n} \mid \Psi_k, \Psi_{k-1}, \cdots) = p(\Psi_{k+n} \mid \Psi_k). \tag{3.206}$$

This is easily shown by induction. Observe that it is true for $n=1$, by the definition of Markov chains (3.78). We can assume that it is true for some n and show it is true for $n+1$. A fact about conditional probabilities similar to that in (3.33) tells us that

$$\begin{aligned} &p(\Psi_{k+n+1} \mid \Psi_k, \Psi_{k-1}, \cdots) \\ &= \sum_{\Psi_{k+1} \in \Omega_\Psi} p(\Psi_{k+n+1} \mid \Psi_{k+1}, \Psi_k, \Psi_{k-1}, \cdots) p(\Psi_{k+1} \mid \Psi_k, \Psi_{k-1}, \cdots). \end{aligned} \tag{3.207}$$

Since we assume that (3.206) is true for n,

$$p(\Psi_{k+n+1} \mid \Psi_{k+1}, \Psi_k, \Psi_{k-1}, \cdots) = p(\Psi_{k+n+1} \mid \Psi_{k+1}). \tag{3.208}$$

It is also therefore true that

$$p(\Psi_{k+n+1} \mid \Psi_{k+1}, \Psi_k, \Psi_{k-1}, \cdots) = p(\Psi_{k+n+1} \mid \Psi_{k+1}, \Psi_k). \tag{3.209}$$

Furthermore, from the definition of Markov chains,

$$p(\Psi_{k+1} \mid \Psi_k, \Psi_{k-1}, \cdots) = p(\Psi_{k+1} \mid \Psi_k). \tag{3.210}$$

Substituting (3.209) and (3.210) into (3.207) we get

$$p(\Psi_{k+n+1} \mid \Psi_k, \Psi_{k-1}, \cdots) = \sum_{\Psi_{k+1} \in \Omega_\Psi} p(\Psi_{k+n+1} \mid \Psi_{k+1}, \Psi_k) p(\Psi_{k+1} \mid \Psi_k). \tag{3.211}$$

Using the same fact about conditional probabilities (3.33) we can eliminate the summation to get

$$p(\Psi_{k+n+1} \mid \Psi_k, \Psi_{k-1}, \cdots) = p(\Psi_{k+n+1} \mid \Psi_k), \tag{3.212}$$

which shows that (3.79) is valid for $n+1$.

Exercise 3-14. Multiplying both sides of (3.83) by z^{-k} and summing from $k=0$ to $k=\infty$,

$$\sum_{k=0}^{\infty} p_{k+1}(j) z^{-k} = \sum_{i \in \Omega_\Psi} p(j \mid i) \sum_{k=0}^{\infty} p_k(i) z^{-k} .$$

Changing variables and letting $m=k+1$,

$$\sum_{m=1}^{\infty} p_m(j) z^{-m+1} = \sum_{i \in \Omega_\Psi} p(j \mid i) P_i(z)$$

or

$$z(P_j(z) - p_0(j)) = \sum_{i \in \Omega_\Psi} p(j \mid i) P_i(z)$$

Exercise 3-15. We have

$$\begin{aligned} f_N &= \sum_{k=-\infty}^{\infty} k q_k(N) \\ &= \sum_{k=-\infty}^{\infty} k q_k(N) z^{-k} \Big|_{z=1} . \end{aligned} \tag{3.213}$$

But the latter summation can be evaluated using a derivative,

$$\frac{\partial}{\partial z} Q_N(z) = \sum_{k=-\infty}^{\infty} (-k) q_k(N) z^{-k-1} = -z^{-1} \sum_{k=-\infty}^{\infty} k q_k(N) z^{-k} . \tag{3.214}$$

The result follows immediately.

Exercise 3-16.

a. Given the power series

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} \tag{3.215}$$

and differentiating it once

$$e^a = \frac{1}{a} \sum_{k=1}^{\infty} \frac{ka^k}{k!} \tag{3.216}$$

and differentiating it twice

$$e^a = \frac{1}{a^2} \left[\sum_{k=1}^{\infty} \frac{k^2 a^k}{k!} - \sum_{k=1}^{\infty} \frac{ka^k}{k!} \right]. \tag{3.217}$$

The moments follow immediately.

b. The moment generating function is

$$\Phi_N(s) = e^{-a} \sum_{k=-\infty}^{\infty} \frac{(ae^s)^k}{k!} = e^{ae^s - a} \tag{3.218}$$

and taking the logarithm we get (3.108).

Exercise 3-17. For these initial conditions, we get

$$q_k(t_0) = 1 \tag{3.219}$$

and the Laplace transform becomes

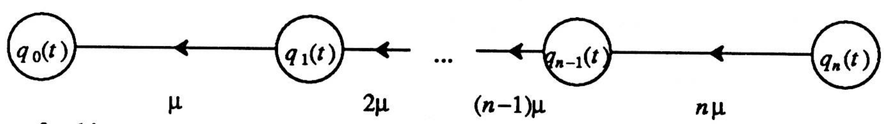
$$\begin{aligned} sQ_j(s) + \lambda Q_j(s) &= \lambda Q_{j-1}(s), \; j \neq k \\ sQ_k(s) - q_k(t_0)e^{-st_0} + \lambda Q_k(s) &= \lambda Q_{k-1}(s). \end{aligned} \tag{3.220}$$

By iteration, we can establish that $Q_j(s) = 0$ for $j < k$ and for $j \geq k$

$$Q_j(s) = \frac{\lambda^{j-k}}{(s+\lambda)^{j-k+1}} e^{-st_0}. \tag{3.221}$$

The result follows immediately by taking the inverse Laplace transform.

Exercise 3-18. The state transition diagram is shown in the following figure:



The equations become for this case

$$\begin{aligned} &\frac{dq_n(t)}{dt} + n\mu q_n(t) = 0 \\ &\frac{dq_j(t)}{dt} + j\mu q_j(t) = (j+1)\mu q_{j+1}(t), \; 0 \leq j < n \end{aligned} \tag{3.222}$$

with initial condition

$$q_j(0) = \begin{cases} 0, & 0 \leq j < n \\ 1, & j = n \end{cases}. \tag{3.223}$$

Taking the Laplace transform, we get

$$\begin{aligned} sQ_n(s) - q_n(0) + n\mu Q_n(s) &= 0 \\ sQ_j(s) + j\mu Q_j(s) &= (j+1)\mu Q_{j+1}(s), \; 0 \leq j < n. \end{aligned} \tag{3.224}$$

It follows that

$$\begin{aligned} q_n(t) &= e^{-n\mu t} \\ q_j(t) &= (j+1)\mu e^{-j\mu t} * q_{j+1}(t) \end{aligned} \tag{3.225}$$

and the reader can verify by induction that (3.110) is valid.