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Mathematics Applied to Mechanics

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Dedicated to
Aldwyth and Elwyn,
Joy,
Jennifer and Siân

Preface

This book provides an introduction to Newtonian Mechanics suitable for students at universities and polytechnics, and also for those specializing in the subject in their final years at school. A primary emphasis has been placed throughout on mathematical models and the equations of motion; a secondary, but important, role being played by the conservation principles. The reader is thereby encouraged to apply to all problems a consistent approach which involves the choice of a frame of reference, the representation of a body's position, velocity and acceleration, the representation of all forces acting on a body, and the application of mathematical techniques to solve the governing equation of motion subject to given initial conditions. It has been assumed that the reader has, or is acquiring, a knowledge of relevant topics in algebra, calculus, geometry and trigonometry. Vectors and differential equations are used throughout the book, and numerical methods are referred to on appropriate occasions.

The reader is urged to have pen and paper at hand. It is good practice to work out on paper the details of any calculation or manipulation. The reading of a mathematical textbook should be different from the reading of a novel.

Graded sets of exercises are provided at the ends of chapters. There is no substitute for the hard work involved in the attempting of exercises. The reader is invited frequently to make a complete analysis of a particular problem rather than to provide information about one aspect, as is usually the case in examination questions. The many worked examples in the text encourage this approach to problems. Examinations do have to be passed, however, so questions set in the appropriate examination held in previous years should be attempted to gain experience.

Three books that may widen horizons and suggest project work are: *Animal Mechanics* by R. McNeill Alexander, Sidgwick and Jackson (1968); *The Physics of Ball Games* by C. B. Daish, The English Universities Press (1972); *Discrete Models* by Donald Greenspan, Addison-Wesley (1973).

Articles that appear in the monthly publication, *The American Journal of Physics*, provide another source of stimulating material.

This book was planned in collaboration with Mr W. T. Blackburn, Principal Lecturer in Mathematics at Dundee College of Education. His untimely death in 1974 meant that the writing of the book fell to me, but I should like to acknowledge the benefits of his careful analysis of my preliminary ideas.

It gives me much pleasure to express my thanks to Miss R. A. Dudgeon and Mrs A. J. Fraser for preparing the typescript.

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Introduction

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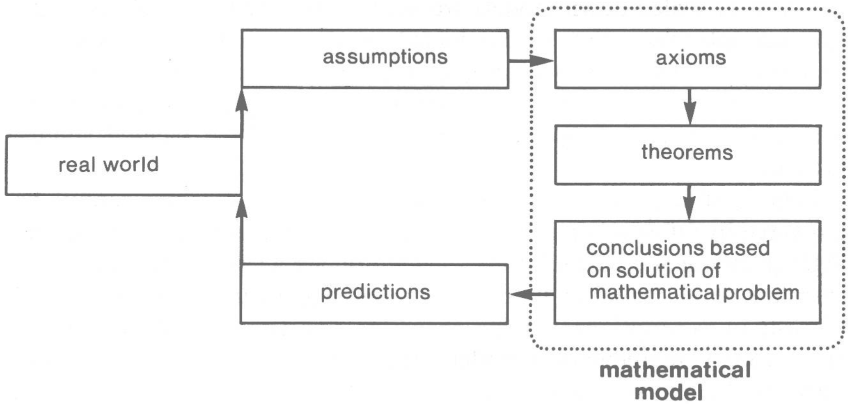
1.1 Mathematical models

When an aeroplane is being designed and developed, many tests and experiments are performed before the prototype takes to the air. Sometimes full-scale or scaled-down models are used in these tests and experiments. For example, tests may be conducted on the strength of parts of the framework subjected to vibrations, or wind-tunnel experiments may be performed to check the effects of the shape of the fuselage at different speeds. The development of a ship, e.g. a supertanker or a lifeboat, includes similar procedures.

Some problems in the physical sciences and engineering cannot be answered by full-scale or scaled-down testing. Weather forecasting, space-flight planning and nuclear-power generation are a few examples. In these cases the development of an abstract model may be useful. Such a model consists of assumptions, axioms, theorems, conclusions and predictions. It is called a "mathematical model". These models are of importance not only in the physical sciences and engineering, but also in business, economics, and the biological and medical sciences.

In geometry, a point, a straight line, and a plane are abstract ideas. They represent a dot on a page, a line drawn on a page with the aid of a ruler, and the page itself. To describe the shape and position of objects in the real world, it is convenient to work with an abstract (euclidean) space that has particular properties. The space consists of a set of elements, called *points*. The distance between two points is defined. Straight lines are collections of points having specified properties. The angle between two intersecting straight lines is defined. The ideas of distance and angle are fundamental. Their development into a mathematical theory is studied under the heading of "Euclidean Geometry" [see J. Hunter, *Analytic Geometry and Vectors*, Blackie/Chambers]. The predictions of the theory may lead to interesting results in the real world, and situations in the real world may suggest the development of theorems. These links between the real world and the abstract mathematical model are important, particularly their two-way nature. A word of warning should be given. Some illustrations may be misleading and confusing in our study of geometry [see E. A. Maxwell, *Fallacies in Mathematics*, p. 22, Cambridge University Press].

Let us consider the extremely complicated problem of describing the motion of the Earth around the Sun. To make any progress we attempt to develop a mathematical model. The model must be sufficiently simple for us to be able to solve the resulting abstract problem (an idealization of the real problem) with the mathematics available to us. This restriction on the model is always present. It depends on the development of mathematical techniques (e.g. algebra, geometry, calculus, numerical analysis) and not only on *our* knowledge of these techniques. Conversely we must not accept an oversimplified mathematical model. Let us examine the constituent parts of a mathematical model in relation to this problem.



Assumptions	Represent the Earth and the Sun by two points having given properties (mass, initial position, initial velocity). Ignore all other bodies (planets, stars, etc.).
Axioms	Euclidean space Newton's Laws of Motion Newton's Law of Gravitation
Theorems, conclusions	Apply the axioms and solve the resulting mathematical equations (exactly or approximately).
Predictions	Make predictions based on the mathematical model (e.g. the Earth moves round the Sun once a year) and carry out experiments in the real world to test the predictions.

We have assumed that the principal factor in this problem is the gravitational force acting between the Earth and the Sun, and we have ignored all other factors (e.g. the finite dimensions of the Earth and the Sun, the influence of the Moon and the planets). Our model will not predict the spinning of the Earth about its axis, but it will predict that the Earth moves round the Sun once a year.

As a second example let us consider a pendulum made from a ball and a length of string. It is worth while to list some of the possible assumptions. We may suppose that:

(i) The ball is represented by a point having mass (ignoring the shape and dimensions of the ball).
(ii) The string has negligible mass and is of constant length (ignoring the mass of the string and any variations in its length).
(iii) The string is attached to a rigid support situated near the surface of the Earth.
(iv) The only applied force is a constant gravitational force (ignoring the variations of the Earth's gravitational force with height and the presence of other gravitational bodies, e.g. Sun, Moon, planets).
(v) The Earth is at rest (ignoring the effects of the rotation of the Earth).
(vi) The pendulum is in a vacuum (ignoring air resistance).

We shall examine this problem under these six assumptions in chapter 7. Another way to look at the problem is to list the factors that may influence the motion of the pendulum. We may ask ourselves the following questions:

(i) How important are the shape and the size of the ball?
(ii) How is the ball fixed to the string?
(iii) What supports the string? Can the support move?
(iv) Does the string extend?
(v) Does the string have the characteristics of a thread or of a rope? How does the mass of the string compare with that of the ball?
(vi) Is air resistance important?
(vii) What applied forces are acting on the pendulum?

When setting up a mathematical model of a real-world problem, we commonly have to decide which factor is most important and then ignore the other factors. If we were to attempt to incorporate all or most of the factors into our mathematical model, the governing mathematical equations would be so complicated that we would be unable to find even an approximate solution. We expect the predictions based on the mathematical model to be in approximate agreement with the results of experiments in the real world. If the agreement between theory and experiment (carefully executed) is not satisfactory, we must examine our assumptions and our axioms. We may single out another factor as being possibly the most important factor. Alternatively, we may try to incorporate two factors into our mathematical model. Sometimes we may find that the axioms used were inappropriate, e.g. we may have to replace Newton's laws by Einstein's axioms of relativity.

It is our task to discuss in this book

(*a*) the setting up of mathematical models;

(*b*) the statement and application of Newton's laws of motion and Newton's law of gravitation;
(*c*) the derivation of mathematical equations and the development of techniques of solution;
(*d*) the design of experiments to test the conclusions.

We shall devote the main effort to discussions of mathematical models (items *b* and *c*), but we shall attempt to retain an awareness of the connections between the real-world problem and the mathematical model under discussion (items *a* and *d*).

1.2 Coordinate systems and frames of reference

We are interested in the positions and motions of bodies, e.g. an aeroplane or a motor-car. We shall represent a body in an abstract way by means of a collection of points in a three-dimensional euclidean space [see J. Hunter, *Analytic Geometry and Vectors*, Blackie/Chambers]. Each point in a three-dimensional euclidean space is specified by an ordered set of three real numbers, these being the coordinates of the point; and the distance between points obeys Pythagoras's theorem. Each point has a unique set of coordinates relative to a given coordinate system. A coordinate system consists of a chosen point, called the *origin* and represented by $(0, 0, 0)$, and surfaces, called *coordinate surfaces*, on which a given coordinate is held fixed and the other two coordinates are allowed to vary. The simplest example is a cartesian coordinate system. In this case, a general point P has the coordinates (x, y, z), and the coordinate surfaces are planes intersecting at right angles. The coordinate axes are the lines of intersection of the coordinate surfaces corresponding to $x = 0$, $y = 0$, $z = 0$. The coordinate surface corresponding to $x = 0$ is the set of points $\{(0, y, z): -\infty < y < \infty, -\infty < z < \infty\}$. Similarly we define the coordinate surfaces corresponding to $y = 0, z = 0$ respectively. In this way, we see that the x-axis is the set of points $\{(x, 0, 0): -\infty < x < \infty\}$, the y-axis is the set of points $\{(0, y, 0): -\infty < y < \infty\}$ and the z-axis is the set of points $\{(0, 0, z): -\infty < z < \infty\}$. The origin is the point of intersection of the three coordinate axes.

Let the distance between two points A and B be $d(A, B)$. Then

$$d(A, B) = \{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2\}^{1/2}$$

where the coordinates of A are (x_A, y_A, z_A) and the coordinates of B are (x_B, y_B, z_B). Sometimes it is convenient to work with part of the three-dimensional space. Particular cases are a plane and a straight line. If a plane is appropriate, we usually use the coordinate surface corresponding to $z = 0$, which contains the x-axis and the y-axis. In this case the distance between two points A, B in the plane is

$$d(A, B) = \{(x_A - x_B)^2 + (y_A - y_B)^2\}^{1/2}$$

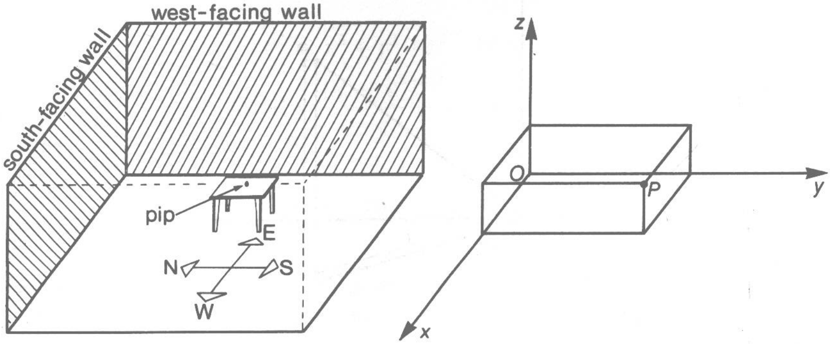
and we say that the coordinates of A are (x_A, y_A). The name of the plane is the xy-plane. If a straight line is appropriate, we may take the x-axis (or one of the other axes). The distance between two points A, B on the line is

$$d(A,B) = \{(x_A - x_B)^2\}^{1/2} = |x_A - x_B|$$
$$= \begin{cases} x_A - x_B & \text{if} \quad x_B \leqslant x_A \\ x_B - x_A & \text{if} \quad x_A \leqslant x_B \end{cases}$$

and we say that the coordinate of A is x_A.

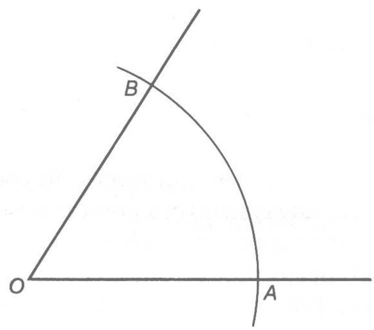
In a real-world situation, distances are measured by making comparisons with measuring rods or tape measures. The length of a measuring aid is standardized in terms of a chosen system of units. We shall use SI units (International System of Units) throughout the book. Our unit of length will be the metre (the SI symbol being m). This has been standardized by international agreement in terms of the wavelength of radiation emitted by the krypton-86 atom under specified conditions [see *Quantities, Units and Symbols*, a report by The Symbols Committee of the Royal Society, 1975]. In common terms, the height of a three- to four-year-old child is approximately one metre, and the height of a doorway in a modern house is approximately two metres.

Once we can measure distance, we can specify the coordinates (relative to a chosen cartesian coordinate system) of a point representing the position of a body. Consider a pip of an apple on a table in a room. After measuring, we may say that the pip is (approximately) one metre from the west-facing wall, two metres from the south-facing wall, and three-quarters of a metre from the floor. We are ignoring here the dimensions of the pip itself. (To be more precise we may say that the pip is between 1·000 and 1·003 metres from the west-facing wall, between 1·999 and 2·001 metres from the south-facing wall, and between 0·749 and 0·750 metres from the floor.) If we choose a



cartesian coordinate system such that the xy-plane represents the floor, the yz-plane represents the west-facing wall and the zx-plane represents the south-facing wall, then we can say that the point with coordinates $(1, 2, \frac{3}{4})$ represents the pip of the apple. We have adopted a scale of measurement in doing this. If we had chosen a different scale, we could have said that the point with coordinates (100, 200, 75) represented the pip of the apple. We see that a cartesian coordinate system is based on the three coordinate axes and a unit of measurement.

Let us consider briefly the measurement of angle. Our measuring instrument is the protractor, and our unit of measurement is the radian (the SI symbol being rad). Consider the acute angle AOB in the figure below. We have an angle of one radian when the length of the circular arc AB equals the radius of the arc. We may note that the degree is an alternative unit of angular measurement, and that 360 degrees are equivalent to 2π radians.



It is convenient in some problems to introduce special systems of coordinates, e.g. cylindrical polar coordinates or spherical polar coor-