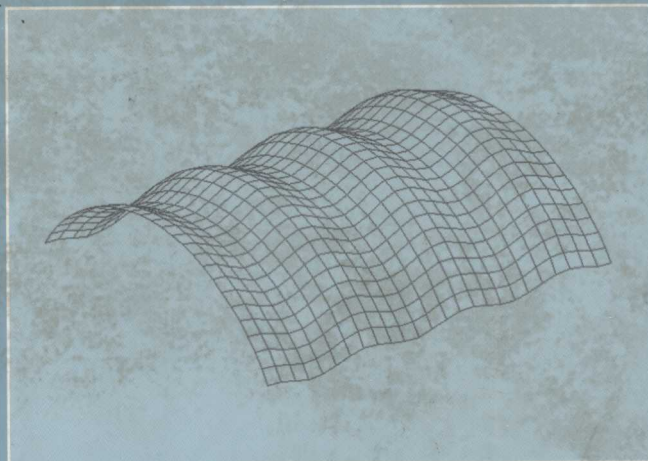
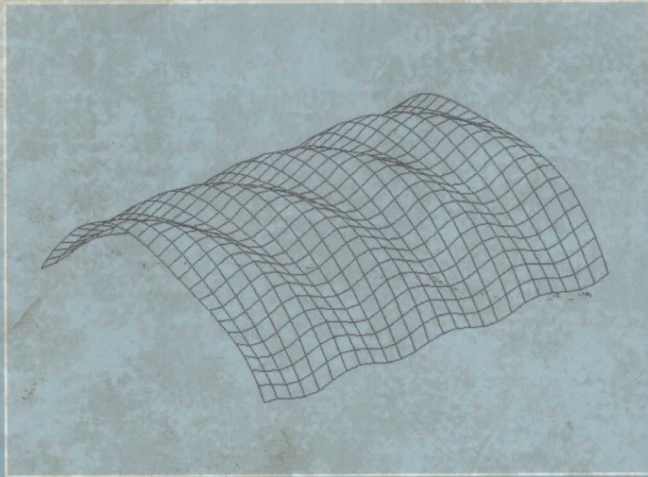


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# MECHANICS OF Composite Structures

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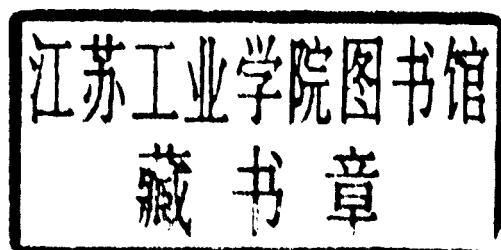
# **MECHANICS OF COMPOSITE STRUCTURES**

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## Preface

The increased use of composites in aerospace, land, and marine applications has resulted in a growing demand for engineers versed in the design of structures made of fiber-reinforced composite materials. To satisfy this demand, and to introduce engineers to the subject of composites, numerous excellent texts have been published dealing with the mechanics of composites. These texts deal with those fundamental aspects needed by engineers new to the subject. Our book addresses topics not generally covered by existing texts but that are necessary for designing practical structures. Among the topics in this book of special interest to the designer, but that usually are not included in standard texts, are stress-strain relationships for a wide range of anisotropic materials; bending, buckling, and vibration of plates; bending, torsion, buckling, and vibration of solid as well as thin-walled beams; shells; hygrothermal stresses and strains; and finite element formulation. The material is presented in sufficient detail to enable the reader to follow the developments leading to the final results. The expressions resulting from the analyses are either readily usable or can be translated into a computer algorithm. Thus, the book should be useful to students and researchers wishing to acquire knowledge of some of the advanced concepts of the mechanics of composites as well as to engineers engaged in the design of structures made of composite materials.

The emphasis is on analyses built on fundamental concepts that are applicable to a variety of structural design problems. In presenting the material we have strived to follow the outline commonly used in texts dealing with the analysis of structures made of isotropic materials. We have consciously omitted empirical approaches. Test results are certainly of value to the engineer. However, for composites, these mostly apply only under specific circumstances and cannot readily be generalized to different materials and different applications. We have included material properties data to help the designer perform calculations without the need to search the literature.

The book is self-contained. Nevertheless, the reader will find it helpful to have a background in mechanics and in composites and some knowledge of differential

equations and matrix algebra. We have made an effort to keep the notation as uniform as practicable and reasonably consistent with accepted usage. The principal symbols are summarized in a list of symbols.

We are grateful to Professor István Hegedűs for his constructive comments. We thank Dr. Rita Kiss, Gabriella Tarján, and Anikó Pluzsik for proofreading portions of the manuscript, Gabriella Tarján for preparing the illustrations, and Eric Allison and Sarah Brennan for their help in compiling the index.

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## List of Symbols

We have used, wherever possible, notation standard in elasticity, structural analysis, and composite materials. We tried to avoid duplication, although there is some repetition of those symbols that are used only locally. In the following list we have not included those symbols that pertain only to the local discussion. Below, we give a verbal description of each symbol and, when appropriate, the number of the equation in which the symbol is first used.

### Latin letters

$A$	area
$A^{\text{iso}}$	tensile stiffness of an isotropic laminate (Eq. 3.42)
$[A], A_{ij}$	tensile stiffness of a laminate (Eqs. 3.18, 3.19)
$[a], a_{ij}$	inverse of the $[A]$ matrix for symmetric laminates (Eq. 3.29)
$[B], B_{ij}$	stiffness of a laminate (Eqs. 3.18, 3.19)
$[C], C_{ij}$	3D stiffness matrix in the $x_1, x_2, x_3$ coordinate system (Eq. 2.22)
$[\bar{C}], \bar{C}_{ij}$	3D stiffness matrix in the $x, y, z$ coordinate system (Eq. 2.19)
$c$	moisture concentration (Eq. 2.154); core thickness (Fig. 5.2)
$[D], D_{ij}$	bending stiffness of a laminate (Eqs. 3.18, 3.19)
$[D]^*, D_{ij}^*$	reduced bending stiffness of a laminate (Eq. 4.1)
$D^{\text{iso}}$	bending stiffness of an isotropic laminate (Eq. 3.42)
$D, \bar{D}, \widehat{D}$	parameters (Table 6.2, page 222, Eq. 6.157)
$[d], d_{ij}$	inverse of the $[D]$ matrix for symmetrical laminates (Eq. 3.30)
$d, d^t, d^b$	distances for sandwich plates (Fig. 5.2)
$E_1, E_2, E_3$	Young's moduli in the $x_1, x_2, x_3$ coordinate system (Table 2.5)
$[E]$	stiffness matrix in the FE calculation (Eq. 9.4)
$\widehat{EA}$	tensile stiffness of a beam (Eq. 6.8)
$\widehat{EI}$	bending stiffness of a beam (Eq. 6.8)
$\widehat{EI}_\omega$	warping stiffness of a beam (Eq. 6.244)
$F_i, F_{ij}$	strength parameters in the quadratic failure criterion (Eq. 10.2)

$f_{ij}$	constants in the quadratic failure criterion (Eq. 10.25)
$f, f_{ij}$	frequency (Eq. 4.190)
$f_x, f_y, f_z$	body forces per unit volume (Eq. 2.13)
$G_{23}, G_{13}, G_{12}$	shear moduli in the $x_1, x_2, x_3$ coordinate system (Table 2.5)
$\widehat{GI}_t$	torsional stiffness of a beam (Eq. 6.8)
$h$	plate thickness
$h_b, h_t$	distances of the bottom and top surfaces of a plate from the reference plane (Eq. 3.9)
$i_\omega$	polar radius of gyration (Eq. 6.340)
$[J]$	inverse of the material stiffness matrix $[E]$ (Eq. 9.16)
$K$	number of layers in a laminate; number of wall segments; stiffness parameter of a plate (Eq. 4.153)
$\tilde{k}$	rotational spring constant (Eq. 4.149)
$k$	equivalent length factor (Eq. 6.340)
$L_x, L_y$	dimensions of a plate
$L$	length; number of cells in a multicell beam (Eq. 6.222)
$L_i, L_i^f$	load and failure load (Eq. 10.42)
$l_x, l_x^o$	half buckling length (Eq. 4.142), half buckling length corresponding to the lowest buckling load of a long plate (Eq. 4.173)
$M_x, M_y, M_{xy}$	bending and twist moments per unit length acting on a laminate (Eq. 3.9)
$M_x^{ht}, M_y^{ht}, M_{xy}^{ht}$	hygrothermal moments per unit length (Eq. 4.247)
$\widehat{M}_y, \widehat{M}_z$	bending moments acting on a beam (Fig. 6.2)
$\widehat{M}_\omega$	bimoment acting on a beam (Eq. 6.232)
$N_x, N_y, N_{xy}$	in-plane forces per unit length acting on a laminate (Eq. 3.9)
$N_{x0}, N_{y0}, N_{xy0}$	in-plane compressive forces per unit length (Eq. 4.109)
$N_x^{ht}, N_y^{ht}, N_{xy}^{ht}$	hygrothermal forces per unit length (Eq. 4.246)
$N_{x,cr}$	buckling load of a uniaxially loaded plate (Eq. 4.141)
$\widehat{N}$	axial force acting on a beam (Fig. 6.2)
$\widehat{N}_{cr}, \widehat{N}_{cr}^B$	buckling load and buckling load due to bending deformation (Eq. 6.337)
$\widehat{N}_{cry}, \widehat{N}_{crz}$	buckling load in the $x$ - $z$ and $x$ - $y$ planes, respectively (Eqs. 6.337, 7.110)
$\widehat{N}_{cr\psi}$	buckling load under torsional buckling (Eqs. 6.337, 7.110)
$[P], [\bar{P}]$	stiffness matrix of a beam (Eqs. 6.2, 6.250). Without bar refers to the centroid; with bar to an arbitrarily chosen coordinate system
$p$	transverse load per unit area; distance between the origin and the tangent of the wall of a beam (Eq. 6.190)
$p_x, p_y, p_z$	axial and transverse loads (per unit length) acting on a beam (Fig. 6.1); surface forces per unit area (Eq. 2.166)
$[Q], Q_{ij}$	2D plane-stress stiffness matrix in the $x_1, x_2$ coordinate system (Eq. 2.134)

$[\bar{Q}], \bar{Q}_{ij}$	2D plane-stress stiffness matrix in the $x, y$ coordinate system (Eq. 2.126)
$\hat{Q}_{cr}$	buckling load resulting in lateral buckling (Eq. 6.359)
$q$	shear flow (Eq. 6.189).
$R$	stiffness parameter (Eq. 3.46)
$\tilde{R}$	stress ratio (Eq. 10.42)
$R_x, R_y, R_{xy}$	radii of curvatures of a shell (Eq. 8.1)
$[R], R_{ij}$	compliance matrix under plane-strain condition in the $x_1, x_2$ coordinate system (Eq. 2.79)
$[\bar{R}], \bar{R}_{ij}$	compliance matrix under plane-strain condition in the $x, y$ coordinate system (Eq. 2.65)
$[S], S_{ij}$	3D compliance matrix in the $x_1, x_2, x_3$ coordinate system (Eq. 2.23)
$[\bar{S}], \bar{S}_{ij}$	3D compliance matrix in the $x, y, z$ coordinate system (Eq. 2.21)
$\hat{S}_{ij}$	shear stiffness of a beam, $i, j = z, y, \omega$ (Eqs. 7.13, 7.36)
$\tilde{S}_{ij}$	shear stiffness of a plate, $i, j = 1, 2$ (Eq. 5.15)
$\hat{s}_{ij}$	shear compliance of a beam, $i, j = z, y, \omega$ (Eq. 7.38)
$s_1^+, s_2^+, s_3^+$	tensile strengths (Eq. 10.13)
$s_1^-, s_2^-, s_3^-$	compression strengths (Eq. 10.13)
$s_{23}, s_{13}, s_{12}$	shear strengths (Eq. 10.15)
$\hat{T}$	torque acting on a beam (Fig. 6.2)
$\hat{T}_\omega$	restrained warping-induced torque (Eq. 6.235)
$\hat{T}_{sv}$	Saint-Venant torque (Eq. 6.239)
$[T_\sigma]$	2D stress transformation matrix (Eq. 2.182)
$[\hat{T}_\sigma]$	3D stress transformation matrix (Eq. 2.179)
$[T_\epsilon]$	2D strain transformation matrix (Eq. 2.188)
$[\hat{T}_\epsilon]$	3D strain transformation matrix (Eq. 2.185)
$t$	torque load acting on a beam (Fig. 6.1)
$t^t, t^b$	thicknesses of the top and bottom facesheets (Eq. 5.26)
$U$	strain energy (Eq. 2.200)
$U$	displacement in the $x$ direction; varies with the $x$ and $y$ coordinates only (Eq. 2.50)
$u$	displacement in the $x$ direction
$u^o$	displacement of the reference surface in the $x$ direction
$u_1, u_2, u_3$	displacements in the $x_1, x_2$ , and $x_3$ direction
$V$	displacement in the $y$ direction; varies with the $x$ and $y$ coordinates only (Eq. 2.51)
$V_f, V_m, V_v$	volume of fibers, matrix, and void
$V_x, V_y$	out-of-plane shear forces per unit length (Eq. 3.10)
$\hat{V}_y, \hat{V}_z$	transverse shear forces acting on a beam (Fig. 6.2)
$v$	displacement in the $y$ direction
$v^o$	displacement of the reference surface in the $y$ direction
$v_f, v_m, v_v$	volume fraction of fibers, matrix, and void



$W$	displacement in the $z$ direction; varies with the $x$ and $y$ coordinates only (Eq. 2.52)
$[W], [\bar{W}]$	compliance matrix of a beam (Eq. 6.17). No bar refers to the centroid; bar to an arbitrarily chosen coordinate system
$w$	deflection in the $z$ direction
$\tilde{w}$	maximum deflection in the $z$ direction (Eq. 4.29)
$w^o$	deflection of the reference surface in the $z$ direction
$w^B, w^S$	deflections due to bending and shear deformations (Eq. 7.85)
$y_c, z_c$	coordinates of the centroid of a beam (Eqs. 6.54, 6.73)
$y_{sc}, z_{sc}$	coordinates of the shear center of a beam (Eq. 6.311)
$z_k, z_{k-1}$	coordinates of the top and bottom surfaces of the $k$ th ply in a laminate (Eq. 3.20)

### Greek letters

$\alpha$	parameter describing shear deformation (Eq. 7.253)
$\alpha_i$	parameter describing shear deformation, $i = w, \psi, N, \omega$ (Eq. 7.244)
$[\alpha], \alpha_{ij}$	compliance matrix of a laminate (Eq. 3.23)
$\alpha, \beta$	parameters describing buckled shape of a shell (Eq. 8.78)
$\hat{\alpha}_{ij}$	compliances for closed-section beams (Eq. 6.156)
$\tilde{\alpha}_i, \tilde{\alpha}_{ij}$	thermal expansion coefficients (Eqs. 2.153, 2.158)
$\beta, \lambda$	parameters in the displacements of a cylinder (Eq. 8.30)
$[\beta], \beta_{ij}$	compliance matrix of a laminate (Eq. 3.23)
$\bar{\beta}_{ij}$	compliance of symmetrical cross-section beams (Table 6.2)
$\hat{\beta}_{ij}$	compliance of closed-section beams (Eq. 6.156)
$\tilde{\beta}_i, \tilde{\beta}_{ij}$	moisture expansion coefficients in the $x, y, z$ directions (Eqs. 2.154, 2.159)
$\beta_1$	property of the cross section (Eq. 6.360)
$\gamma_y, \gamma_z$	shear strain in a beam in the $x$ - $y$ and $x$ - $z$ planes (Eq. 7.2)
$\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$	engineering shear strain in the $x, y, z$ coordinate system (Eq. 2.9)
$\gamma_{23}, \gamma_{13}, \gamma_{12}$	engineering shear strain in the $x_1, x_2, x_3$ coordinate system
$\Delta h$	change in thickness (Eq. 4.282)
$\Delta T$	temperature change (Eq. 2.153)
$[\delta], \delta_{ij}$	compliance matrix of a laminate (Eq. 3.23)
$\hat{\delta}_{ij}$	compliance of closed-section beams (Eq. 6.157)
$\bar{\epsilon}_x, \dots$	average strains in a sublaminates (Eq. 9.14)
$\epsilon_x, \epsilon_y, \epsilon_z$	engineering normal strains in the $x, y, z$ coordinate system
$\epsilon_1, \epsilon_2, \epsilon_3$	engineering normal strains in the $x_1, x_2, x_3$ coordinate system
$\epsilon_x^o, \epsilon_y^o, \gamma_{xy}^o$	strains of the reference surface
$\epsilon_x^{o,ht}, \epsilon_y^{o,ht}, \gamma_{xy}^{o,ht}$	hygrothermal strains in a laminate (Eq. 4.250)
$\zeta$	parameter of restraint (Eq. 4.152)
$\Theta$	polar moment of mass (Eq. 6.411)

$\Theta_k$	ply orientation
$\vartheta$	rate of twist (Eq. 6.1)
$\vartheta^B, \vartheta^S$	rate of twist due to bending and shear deformation (Eq. 7.5)
$\kappa_x, \kappa_y, \kappa_{xy}$	curvatures of the reference surface (Eq. 3.8)
$\kappa_x^{ht}, \kappa_y^{ht}, \kappa_{xy}^{ht}$	hygrothermal curvatures of a laminate (Eq. 4.250)
$\lambda, \lambda_{cr}, \lambda_{ij}$	load parameter (Eq. 4.109); buckling load parameter (Eq. 4.121); eigenvalue (Eq. 4.225)
$\mu_{Bi}, \mu_{Gi}, \mu_{Si}$	parameters in the calculation of natural frequencies (Eqs. 6.398, 6.400, 7.203)
$\nu_{ij}$	Poisson's ratio
$\xi, \eta, \zeta$	coordinates attached to the wall of a beam (Fig. 6.13)
$\xi, \xi'$	parameters in the expressions of the buckling loads of plates with rotationally restrained edges (Eq. 4.151)
$\pi_p$	potential energy (Eq. 2.204)
$\rho_x, \rho_y, \rho_z$	radius of curvature in the $y$ - $z$ , $x$ - $z$ , and $x$ - $y$ planes (Eq. 2.45)
$\rho_1, \rho_2, \rho_3$	radius of curvature in the $x_2$ - $x_3$ , $x_1$ - $x_3$ , and $x_1$ - $x_2$ planes (Eq. 2.53)
$\rho_{comp}, \rho_f, \rho_m$	densities of composite, fiber, and matrix
$\rho$	mass per unit area or per unit length
$\sigma_1, \sigma_2, \sigma_3$	normal stresses in the $x_1, x_2, x_3$ coordinate system
$\sigma_x, \sigma_y, \sigma_z$	normal stresses in the $x, y, z$ coordinate system
$\bar{\sigma}$	average stress
$\tau_{23}, \tau_{13}, \tau_{12}$	shear stresses in the $x_1, x_2, x_3$ coordinate system
$\tau_{yz}, \tau_{xz}, \tau_{yx}$	shear stresses in the $x, y, z$ coordinate system
$\chi_{xz}, \chi_{yz}$	rotation of the normal of a plate in the $x$ - $z$ and $x$ - $y$ planes (Eqs. 3.2 and 5.1)
$\chi_y, \chi_z$	rotation of the cross section of a beam in the $x$ - $y$ and $x$ - $z$ planes (Eq. 7.2)
$\psi$	angle of rotation of the cross section about the beam axis (twist) (Fig. 6.3)
$\Psi$	bending stiffness of an unsymmetrical long plate (Eq. 4.52)
$\Omega$	potential energy of the external loads (Eq. 2.203)
$\omega$	circular frequency (Eq. 4.190)
$\omega^B, \omega^S$	circular frequency of a beam due to bending and shear deformation (Eq. 7.198)
$\omega_y, \omega_z$	circular frequency of a freely vibrating beam in the $x$ - $z$ and $x$ - $y$ planes, respectively (Eq. 6.398)
$\omega_\psi$	circular frequency of a freely vibrating beam under torsional vibration (Eq. 6.400)
$\varrho, \tilde{\varrho}, \bar{\varrho}, \hat{\varrho}$	distances between the new and the old reference surfaces (Eqs. 3.47, 6.105, 6.107, A.3)

# Contents

<i>Preface</i>	<i>page xi</i>
<i>List of Symbols</i>	<i>xiii</i>
<b>1 Introduction</b>	<b>1</b>
<b>2 Displacements, Strains, and Stresses</b>	<b>3</b>
2.1 Strain–Displacement Relations	4
2.2 Equilibrium Equations	6
2.3 Stress–Strain Relationships	8
2.3.1 Generally Anisotropic Material	8
2.3.2 Monoclinic Material	11
2.3.3 Orthotropic Material	14
2.3.4 Transversely Isotropic Material	19
2.3.5 Isotropic Material	20
2.4 Plane–Strain Condition	22
2.4.1 Free End – Generally Anisotropic Material	28
2.4.2 Free End – Monoclinic Material	30
2.4.3 Free End – Orthotropic, Transversely Isotropic, or Isotropic Material	34
2.4.4 Built-In Ends – Generally Anisotropic Material	35
2.4.5 Built-In Ends – Monoclinic Material	36
2.4.6 Built-In Ends – Orthotropic, Transversely Isotropic, or Isotropic Material	38
2.5 Plane–Stress Condition	38
2.6 Hygrothermal Strains and Stresses	44
2.6.1 Plane–Strain Condition	47
2.6.2 Plane–Stress Condition	47
2.7 Boundary Conditions	47
2.8 Continuity Conditions	48
2.9 Stress and Strain Transformations	49
2.9.1 Stress Transformation	50
2.9.2 Strain Transformation	52
2.9.3 Transformation of the Stiffness and Compliance Matrices	53

2.10 Strain Energy	55
2.10.1 The Ritz Method	55
2.11 Summary	56
2.11.1 Note on the Compliance and Stiffness Matrices	56
<b>3 Laminated Composites</b>	63
3.1 Laminate Code	63
3.2 Stiffness Matrices of Thin Laminates	65
3.2.1 The Significance of the $[A]$ , $[B]$ , and $[D]$ Stiffness Matrices	72
3.2.2 Stiffness Matrices for Selected Laminates	74
<b>4 Thin Plates</b>	89
4.1 Governing Equations	90
4.1.1 Boundary Conditions	92
4.1.2 Strain Energy	92
4.2 Deflection of Rectangular Plates	93
4.2.1 Pure Bending and In-Plane Loads	93
4.2.2 Long Plates	94
4.2.3 Simply Supported Plates – Symmetrical Layup	100
4.2.4 Plates with Built-In Edges – Orthotropic and Symmetrical Layup	107
4.3 Buckling of Rectangular Plates	112
4.3.1 Simply Supported Plates – Symmetrical Layup	112
4.3.2 Plates with Built-In and Simply Supported Edges – Orthotropic and Symmetrical Layup	118
4.3.3 Plates with One Free Edge – Orthotropic and Symmetrical Layup	124
4.3.4 Plates with Rotationally Restrained Edges – Orthotropic and Symmetrical Layup	127
4.3.5 Long Plates	132
4.4 Free Vibration of Rectangular Plates	141
4.4.1 Long Plates	141
4.4.2 Simply Supported Plates – Symmetrical Layup	144
4.4.3 Plates with Built-In and Simply Supported Edges – Orthotropic and Symmetrical Layup	149
4.5 Hygrothermal Effects	151
4.5.1 Change in Thickness Due to Hygrothermal Effects	161
4.6 Plates with a Circular or an Elliptical Hole	163
4.7 Interlaminar Stresses	166
<b>5 Sandwich Plates</b>	169
5.1 Governing Equations	170
5.1.1 Boundary Conditions	172
5.1.2 Strain Energy	173
5.1.3 Stiffness Matrices of Sandwich Plates	174
5.2 Deflection of Rectangular Sandwich Plates	178
5.2.1 Long Plates	178
5.2.2 Simply Supported Sandwich Plates – Orthotropic and Symmetrical Layup	182

5.3 Buckling of Rectangular Sandwich Plates	185
5.3.1 Long Plates	185
5.3.2 Simply Supported Plates – Orthotropic and Symmetrical Layup	187
5.3.3 Face Wrinkling	190
5.4 Free Vibration of Rectangular Sandwich Plates	196
5.4.1 Long Plates	196
5.4.2 Simply Supported Plates – Orthotropic and Symmetrical Layup	199
<b>6 Beams</b>	203
6.1 Governing Equations	203
6.1.1 Boundary Conditions	205
6.1.2 Stiffness Matrix	205
6.1.3 Compliance Matrix	209
6.1.4 Replacement Stiffnesses	210
6.2 Rectangular, Solid Beams Subjected to Axial Load and Bending	210
6.2.1 Displacements – Symmetrical Layup	211
6.2.2 Displacements – Unsymmetrical Layup	213
6.2.3 Stresses and Strains	214
6.3 Thin-Walled, Open-Section Orthotropic or Symmetrical Cross-Section Beams Subjected to Axial Load and Bending	217
6.3.1 Displacements of T-Beams	217
6.3.2 Displacements of L-Beams	221
6.3.3 Displacements of Arbitrary Cross-Section Beams	226
6.3.4 Stresses and Strains	233
6.4 Thin-Walled, Closed-Section Orthotropic Beams Subjected to Axial Load and Bending	243
6.5 Torsion of Thin-Walled Beams	248
6.5.1 Thin Rectangular Cross Section	248
6.5.2 Open-Section Orthotropic Beams	250
6.5.3 Closed-Section Orthotropic Beams – Single Cell	252
6.5.4 Closed-Section Orthotropic Beams – Multicell	260
6.5.5 Restrained Warping – Open-Section Orthotropic Beams	261
6.5.6 Restrained Warping – Closed-Section Orthotropic Beams	264
6.6 Thin-Walled Beams with Arbitrary Layup Subjected to Axial Load, Bending, and Torsion	265
6.6.1 Displacements of Open- and Closed-Section Beams	267
6.6.2 Stresses and Strains in Open- and Closed-Section Beams	268
6.6.3 Centroid	271
6.6.4 Restrained Warping	271
6.7 Transversely Loaded Thin-Walled Beams	274
6.7.1 Beams with Orthotropic Layup or with Symmetrical Cross Section	276
6.7.2 Beams with Arbitrary Layup	280
6.7.3 Shear Center	283
6.8 Stiffened Thin-Walled Beams	288

6.9 Buckling of Beams	290
6.9.1 Beams Subjected to Axial Load (Flexural–Torsional Buckling)	291
6.9.2 Lateral–Torsional Buckling of Orthotropic Beams with Symmetrical Cross Section	296
6.9.3 Local Buckling	300
6.10 Free Vibration of Beams (Flexural–Torsional Vibration)	306
6.10.1 Doubly Symmetrical Cross Sections	306
6.10.2 Beams with Symmetrical Cross Sections	309
6.10.3 Beams with Unsymmetrical Cross Sections	309
6.11 Summary	312
<b>7 Beams with Shear Deformation</b>	313
7.1 Governing Equations	314
7.1.1 Strain–Displacement Relationships	315
7.1.2 Force–Strain Relationships	315
7.1.3 Equilibrium Equations	320
7.1.4 Summary of Equations	320
7.1.5 Boundary Conditions	321
7.2 Stiffnesses and Compliances of Beams	321
7.2.1 Shear Stiffnesses and Compliances of Thin-Walled Open-Section Beams	322
7.2.2 Shear Stiffnesses and Compliances of Thin-Walled Closed-Section Beams	325
7.2.3 Stiffnesses of Sandwich Beams	326
7.3 Transversely Loaded Beams	329
7.4 Buckling of Beams	334
7.4.1 Axially Loaded Beams with Doubly Symmetrical Cross Sections (Flexural and Torsional Buckling)	335
7.4.2 Axially Loaded Beams with Symmetrical or Unsymmetrical Cross Sections (Flexural–Torsional Buckling)	341
7.4.3 Lateral–Torsional Buckling of Beams with Symmetrical Cross Section	345
7.4.4 Summary	346
7.5 Free Vibration of Beams	347
7.5.1 Beams with Doubly Symmetrical Cross Sections	347
7.5.2 Beams with Symmetrical or Unsymmetrical Cross Sections	356
7.5.3 Summary	359
7.6 Effect of Shear Deformation	359
<b>8 Shells</b>	365
8.1 Shells of Revolution with Axisymmetrical Loading	367
8.2 Cylindrical Shells	368
8.2.1 Membrane Theory	368
8.2.2 Built-In Ends	370
8.2.3 Temperature – Built-In Ends	379
8.3 Springback	380
8.3.1 Springback of Cylindrical Shells	380
8.3.2 Doubly Curved Shells	384

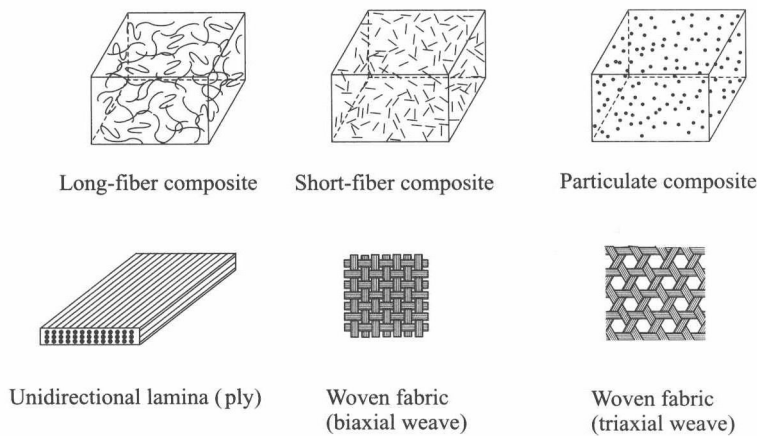
8.4 Buckling of Shells	384
8.4.1 Buckling of Cylinders	387
<b>9 Finite Element Analysis</b>	395
9.1 Three-Dimensional Element	396
9.2 Plate Element	397
9.3 Beam Element	397
9.4 Sublaminates	398
9.4.1 Step 1. Elements of $[J]$ due to In-Plane Stresses	400
9.4.2 Step 2. Elements of $[J]$ due to Out-of-Plane Normal Stresses	403
9.4.3 Step 3. Elements of $[J]$ due to Out-of-Plane Shear Stresses	405
9.4.4 Step 4. The Stiffness Matrix	407
<b>10 Failure Criteria</b>	411
10.1 Quadratic Failure Criterion	413
10.1.1 Orthotropic Material	414
10.1.2 Transversely Isotropic Material	420
10.1.3 Isotropic Material	421
10.1.4 Plane-Strain and Plane-Stress Conditions	422
10.1.5 Proportional Loading – Stress Ratio	423
10.2 “Maximum Stress” Failure Criterion	425
10.3 “Maximum Strain” Failure Criterion	426
10.4 Plate with a Hole or a Notch	430
10.4.1 Plate with a Circular Hole	431
10.4.2 Plate with a Notch	434
10.4.3 Characteristic Length	434
<b>11 Micromechanics</b>	436
11.1 Rule of Mixtures	436
11.1.1 Longitudinal Young Modulus $E_1$	438
11.1.2 Transverse Young Modulus $E_2$	439
11.1.3 Longitudinal Shear Modulus $G_{12}$	439
11.1.4 Transverse Shear Modulus $G_{23}$	440
11.1.5 Longitudinal Poisson Ratio $\nu_{12}$	441
11.1.6 Transverse Poisson Ratio $\nu_{23}$	442
11.1.7 Thermal Expansion Coefficients	443
11.1.8 Moisture Expansion Coefficients	445
11.1.9 Thermal Conductivity	446
11.1.10 Moisture Diffusivity	447
11.1.11 Specific Heat	448
11.2 Modified Rule of Mixtures	448
11.3 Note on the Micromechanics Models	449
<b>Appendix A. Cross-Sectional Properties of Thin-Walled Composite Beams</b>	453
<b>Appendix B. Buckling Loads and Natural Frequencies of Orthotropic Beams with Shear Deformation</b>	461
<b>Appendix C. Typical Material Properties</b>	464
<i>Index</i>	469

**CHAPTER ONE**

**Introduction**

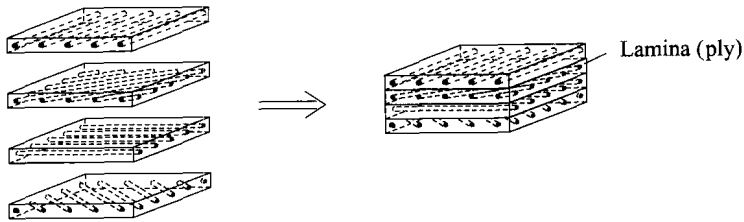
In this book we focus on fiber-reinforced composites composed of fibers embedded in a matrix. The fibers may be short or long, continuous or discontinuous, and may be in one or in multiple directions (Fig. 1.1). Such materials offer advantages over conventional isotropic structural materials such as steel, aluminum, and other types of metal. These advantages include high strength, low weight, and good fatigue and corrosion resistance. In addition, by changing the arrangements of the fibers, the properties of the material can be tailored to meet the requirements of a specific design.

The excellent properties of composites are achieved by the favorable characteristics of the two major constituents, namely the fiber and the matrix. In low-performance composites, the reinforcements, usually in the form of short or chopped fibers (or particles), provide some stiffening but very little strengthening; the load is mainly carried by the matrix. In high-performance composites, continuous fibers provide the desirable stiffness and strength, whereas the matrix provides protection and support for the fibers, and, importantly, helps redistribute the load from broken to adjacent intact fibers.



**Figure 1.1: Composite material systems.**





Laminae forming a laminate                      Laminate

Figure 1.2: Laminated composite.

The arrangement of the fibers in a structure is governed by the structural requirements and by the process used to fabricate the part. Frequently, though not always, composite structures are made of thin layers called laminae or plies. Within each lamina, the fibers may be aligned in the same direction (unidirectional ply, Fig. 1.1) or in different directions. The latter configuration is produced, for example, by weaving the fibers in two or more directions (woven fabric). The lamina may also contain short fibers either oriented in the same direction or distributed randomly. Several laminae are then combined into a laminate to form the desired structure (Fig. 1.2).

The mechanical and thermal behaviors of a structure depend on the properties of the fibers and the matrix and on the amount and orientations of the fibers. In this book, we consider the design steps from micromechanics (which takes into account the fiber and matrix properties) through macromechanics (which treats the properties of the composite) to structural analysis. These steps are illustrated in Figure 1.3 for a structure made of laminated composite.

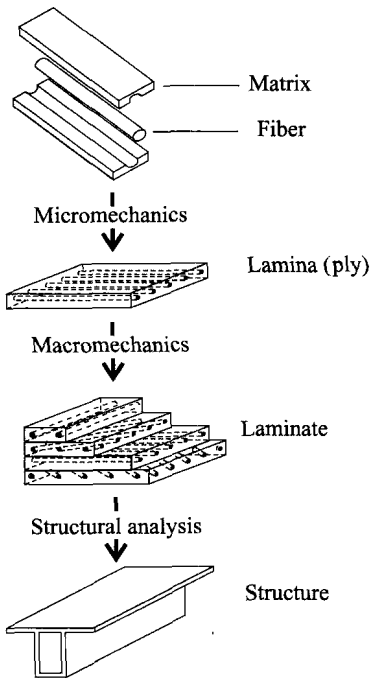


Figure 1.3: The levels of analysis for a structure made of laminated composite.