DEGREE OF APPROXIMATION BY POLYNOMIALS IN THE COMPLEX DOMAIN

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PREFACE

The study of degree of approximation by polynomials in the complex domain is in its second phase. An extensive investigation has been made of the relation of (geometric) degree of convergence by polynomials, approximating functions analytic on a given closed set, to regions of analyticity on the one hand and to regions of uniform convergence on the other hand; the methods and results of this investigation are available in an admirable treatise by Walsh [1935]. The more delicate question of the relation between the behavior (continuity properties, asymptotic conditions, singularities, etc.) of the function on the boundary of the region of convergence, and degree of convergence on the given set of approximating polynomials, is not included in the above treatise. This new problem is the subject treated in the present work.

The analogous question for functions of a single real variable is classical, and has been studied especially by de la Vallée Poussin, Lebesgue, Bernstein, Jackson, and Montel. This involves approximation both by polynomials in the independent variable and by trigonometric sums. Since a polynomial in the complex variable is on the unit circle a trigonometric sum with the real variable chosen as arc length, classical results concerning trigonometric approximation can be applied at once to our present problems. Furthermore the theory of conformal mapping extends the results for the unit circle to a wide variety of regions.

The study of this problem in the complex domain is comparatively recent; the theorem that a function analytic in a Jordan region and continuous in the closed region can be uniformly approximated by polynomials was proved by Walsh in 1926. This theorem along with results and methods concerning regions of analyticity and degree of convergence is a necessary forerunner to a systematic treatment of the present problem.

The appearance of this book does not indicate that the study is completed, far from it, but certain parts of the problem have been solved and the general progress is such that a perspective is now obtainable. In spite of the comparatively few years devoted to an organized study of the question it is impossible to include all of the results in a single treatise. No attempt is made to present an encyclopedic account of the material; the scheme is rather to trace the progress in the study of the problem and to motivate and explain the material evolved. In fact the purpose of the book is not only to present theorems and their proofs but also to stimulate interest in the subject by pointing out the limitations as well as the extent of the existing methods and calling attention to numerous specific problems as yet unsolved.

Many new results are published here for the first time. Much of the material on approximation as measured by a line integral was developed during the preparation of this book; some of the results on Tchebycheff approximation have been extended and refined by the introduction of more powerful methods. Also many of the exercises, stated as theorems, have not previously appeared in the literature; suggestions are given as to their solution in many instances, especially where the methods developed in the text do not apply. At the end of each chapter these exercises are followed by a discussion of open problems.

As the title indicates, the treatment is restricted to approximation by polynomials. Interpolation is incidental and considered only in those cases where it follows naturally from the methods used in studying degree of approximation; many results on interpolating polynomials are included as exercises. Although emphasis is placed on regions bounded by analytic Jordan curves methods which can be readily extended are applied to more general situations. The generalized Lipschitz condition is used to describe the continuity of functions rather than modulus of continuity or generalized derivative. The entire treatment is intended to suggest that the theory studied is a living developing organism rather than an embalmed museum exhibit.

The results and methods of the present treatment can be applied in the study of the analogous theory of approximation to harmonic functions by harmonic polynomials In this connection some partial results have already been obtained by Walsh and the author [1940b].

References to the literature are inserted in the text. Figures in square brackets are dates and indicate particular works on which precise information is given in the Bibliography. In order to facilitate the use of the book, references are made to books rather than to original memoirs whenever possible, preferably to Walsh [1935]. The Bibliography makes no pretensions to completeness; further references to the literature are given for instance by Walsh [1935, 1935a] and by Shohat, Hille, and Walsh [1940].

The author has received invaluable aid in the preparation of this book. Professor J. L. Walsh has not only contributed to practically every page but has been a continual source of inspiration throughout the author's mathematical career. The new research presented in the present work was done largely at the author's own institution, the Georgia School of Technology, but the actual writing of the book was done at Harvard University in the Harvard College Library while the writer was on leave of absence from the Georgia School of Technology. The Julius Rosenwald Fund has been very generous both in grants freeing the author's time for the purpose of writing the book and in arranging for its publication. Mr. Henry Allen Moe has taken an interest in this project far beyond his capacity as a member of the Julius Rosenwald Fund and has encouraged the author throughout this undertaking. The typing of the manuscript and some of the proofreading were done by Miss Jeanne Le Caine. Professor J. H. Curtiss proofread the master copy for lithoprinting. To these sources the author is deeply grateful.

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CONTENTS

	Page
Chapter I. PRELIMINARIES	1
§1. Statement of the Problems; Notation	1
§2. Lipschitz Conditions and Derivatives	8
§3. Exercises	30
§4. Discussion	31
PART I. PROBLEM α	
Chapter II. POLYNOMIAL INEQUALITIES	33
§1. A Polynomial and its Derivarives	33
§2. A Polynomial and its Integral	40
§3. The Lagrange-Hermite Interpolation Formula	43
§4. Geometric Properties of Equipotential Curves	44
§5. Equally Distributed Points	48
§6. Exercises	75
§7. Open Problems	78
Chapter III. TCHEBYCHEFF APPROXIMATION	81
§1. Jackson's Theorem on Trigonometric Approximation	81
§2. Direct Theorems	90
§3. The Faber Polynomials	96
§4. Indirect Theorems	100
§5. The Derivative and Integral of a Function	108
§6. Best Approximation	111
§7. Exercises	112
§8. Open Problems	114

Page

Chapter IV. APPROXIMATION MEASURED BY A LINE INTEGRAL 117
§1. Direct Theorems 117
§2. Indirect Theorems 120
§3. Orthogonal Polynomials 126
§4. Polynomials of Best Approximation 131
§5. Generality of the Weight Function 133
§6. Exercises 135
§7. Discussion 139

PART II. PROBLEM β
Chapter V. PRELIMINARIES 141
§1. An Interpolation Formula and Inequalities 141
§2. An Extended Classification 144
§3. A Polynomial and its Integral 149
§4. Exercises 152
§5. Open Problems 154

Chapter VI. TCHEBYCHEFF APPROXIMATION 156
§1. Direct Theorems 156
§2. Operations with Approximating Sequences . 158
§3. Indirect Theorems 166
§4. Counter Examples 170
§5. Functions with Isolated Singularities ... 172
§6. Polynomials of Best Approximation 176
§7. Exercises 179
§8. Discussion 185

Chapter VII. APPROXIMATION MEASURED BY A LINE INTEGRAL 189
§1. Direct Theorems 189
§2. Indirect Theorems 190
§3. Orthogonal Polynomials 192
§4. Polynomials of Best Approximation 194

Page

Chapter VII. (Continued)
§5. Generality of the Weight Function 195
§6. Exercises 196
§7. Discussion 198

Chapter VIII. SPECIAL CONFIGURATIONS 200
§1. Approximation on $|z| = 1$ by Polynomials in z and $1/z$ 200
§2. Approximation on $-1 \leq z \leq 1$ 208
§3. Trigonometric Approximation 212
§4. Direct Methods on Problem α and Problem β 214
§5. Exercises 218
§6. Discussion 224

Bibliography 226

Chapter I

PRELIMINARIES

§1.1. STATEMENT OF THE PROBLEMS; NOTATION. The domain under consideration throughout the entire treatise is the plane of the complex variable $z = x+iy$. We approximate only on a closed bounded set; we deal thus primarily with the plane of finite points although on occasion we need to study in an auxiliary capacity the plane extended by adjunction of a single point at infinity. The usual definitions of point set theory are assumed, but it is well to recall certain concepts. A <u>Jordan arc</u> is a one to one continuous transform of a line segment, that is, a point set represented by points (x,y) where $x = f(t)$, $y = g(t)$, $0 \leq t \leq 1$, the functions $f(t)$ and $g(t)$ being continuous and admitting a unique solution t for given (x,y). If the functions $f(t)$ and $g(t)$ are analytic and $|f'| + |g'| \neq 0$ the Jordan arc is said to be <u>analytic</u>. A <u>Jordan curve</u> is a one to one continuous transform of a circumference, that is a point set represented by points (x,y) where $x = f(\theta)$, $y = g(\theta)$, $0 \leq \theta \leq 2\pi$, $f(0) = f(2\pi)$, $g(0) = g(2\pi)$, the functions being continuous and admitting a unique solution θ for given (x,y). If the functions are analytic and $|f'| + |g'| \neq 0$ the Jordan curve is said to be <u>analytic</u>. A point set E is <u>connected</u> if any two points of E can be joined by a Jordan arc consisting entirely of points of E. A <u>region</u> is an open connected set. A <u>Jordan region</u> is a region of the finite plane bounded by a Jordan curve. The terms <u>integrable</u> and <u>measurable</u> are in the sense of Lebesgue unless otherwise noted.

A function of the form

$$(1.1.1) \qquad p_n(z) \equiv a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_n, a_{n-1}, \ldots, a_0$ are arbitrary complex constants, is called a polynomial of degree n in z; throughout this book the degree of a polynomial is indicated consistently by its subscript. In (1.1.1) we do not assume $a_n \neq 0$, thus it is clear that a polynomial of degree m, $m < n$, is also a polynomial of degree n; a constant, for example, is a polynomial of every degree. We say that a sequence of polynomials $\{p_n(z)\}$, $n = 0, 1, 2, \ldots$, converges to a function f(z) defined on a set E in the sense of Tchebycheff, or approximates to f(z) on E in the sense of Tchebycheff, if

$$(1.1.2) \qquad |f(z) - p_n(z)| \leq \epsilon_n, \quad z \text{ on } E, \quad \lim_{n \to \infty} \epsilon_n = 0.$$

If

$$\underset{z \text{ on } E}{\text{l.u.b.}} \; |f(z) - p_n(z)| = \epsilon_n,$$

the infinitesimal ϵ_n is a measure of the degree of convergence or the degree of approximation of $p_n(z)$ to f(z) on E in the sense of Tchebycheff, and the sequence $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$ measures the degree of convergence of the sequence $\{p_n(z)\}$ to f(z) on E.

Let E, with boundary C be a closed limited point set in the z-plane and let the function f(z) be defined on E; we use the notation $\overline{C} = E$. Suppose the boundary C of E consists of one or several Jordan curves, and let f(z) be analytic merely in the interior points of E and continuous on E; here approximation on C and E are identical. Problem α denotes the study of the relation between the degree of convergence of certain sequences $p_n(z)$ to f(z) on E on the one hand and the continuity properties of f(z) on C on the other hand. Part I is devoted to a study of Problem α; in particular we study Tchebycheff approximation in Chapter III. This problem has a direct analogue in the real domain, that is for E a segment of the axis of reals; these results,* which are now classical, are applied to the general case considered here. In the complex domain Walsh [1935 pp. 36-39] showed that f(z) can be uniformly approximated on E (including the case of E a Jordan arc); recent contributions to Problem α are due primarily to Curtiss, Sewell, and Walsh and Sewell.

Now let E, with boundary C, be a closed limited set whose complement (with respect to the extended plane) K is connected and regular** in the sense that K possesses a Green's function G(x,y) with pole at infinity; then the function

$$(1.1.3) \qquad w = \phi(z) = e^{G(x,y) + iH(x,y)}, \quad z = x+iy,$$

where H(x,y) is conjugate to G(x,y) in K, maps K conformally, but not necessarily uniformly, on the exterior of the unit circle, γ : $|w| = 1$, in the w-plane so that the points at infinity in the two planes correspond to each other. We denote by C_ρ the image, under the conformal map, in the z-plane of the circle $|\phi(z)| = |w| = \rho > 1$, that is to say C_ρ is the locus $G(x,y) = \log \rho > 0$ in K.

Let f(z) be analytic on E; then there exists a greatest number ρ (finite or infinite) such that f(z) is single valued and analytic at every point interior to C_ρ; if E falls into several disconnected parts the function f(z) defined on its various parts need not consist of the branches of a single monogenic analytic function. If $R < \rho$ and arbitrary there exist polynomials $p_n(z)$, $n = 0, 1, 2, \ldots$, such that

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* See, e.g., Bernstein [1926], Jackson [1930], de la Vallée Poussin [1919].

**See, e.g., Walsh [1935, pp. 65, ff.].

(1.1.4) $$|f(z) - p_n(z)| \leq \frac{M}{R^n}, \quad z \text{ on } E,$$

where M depends on R but not on n or z; when used in an inequality of the form (1.1.4) the letter M with or without subscripts shall hereafter represent a constant which may vary from inequality to inequality but is always independent of n and z. On the other hand there exist no polynomials $p_n(z)$ such that (1.1.4) is valid for z on E with $R > \rho$. A sequence $p_n(z)$, satisfying (1.1.4) for every $R < \rho$ and a suitable M is said to converge to f(z) on E maximally, or with the greatest geometric degree of convergence. Conversely if f(z) is defined on E and if for every n and every $R < \rho$ there exists $p_n(z)$ such that (1.1.4) is valid, then the function f(z) is analytic in the interior of C_ρ. This type of convergence has been fully investigated [Walsh, 1935] and the relation between degree of convergence and regions of analyticity thoroughly studied.

We consider in Part II a more delicate problem whose solution makes use of both the methods and results of the above study. Let f(z) be given analytic interior to some fixed C_ρ and let its continuity properties on or in the neighborhood of C_ρ be precisely described; we designate by Problem β the study of the relation between the degree of convergence of various sequences $p_n(z)$ to f(z) on E on the one hand and the continuity properties of f(z) on or in a neighborhood of C_ρ on the other hand; Chapter VI is devoted to a detailed study of this problem. More discrimination is required in investigating Problem β than was necessary in the study of the relation between geometric degree of convergence and regions of analyticity. The degree of convergence is not expressed in general in a formula as simple as $\epsilon_n = M/\rho^n$; in fact the right member of (1.1.4) is usually multiplied by a positive or negative power of n, $(M/\rho^n).n^\delta$. Our treatment of the problem amounts to a study of the relation between the number δ and the behavior of f(z).

Special cases of this problem have been studied by Bernstein, Faber, and de la Vallée Poussin, among others; the bulk of the material treated in the present work is of recent origin and is due primarily to Walsh and the author.

For the sake of brevity and exposition we distinguish between two types of results in both Problem α and Problem β. A direct theorem is one in which the properties of $f(z)$ are in the hypothesis and the degree of convergence of $p_n(z)$ to $f(z)$ is the conclusion. An indirect theorem is in the converse direction, that is the degree of convergence of $p_n(z)$ to $f(z)$ is in the hypothesis and the properties of $f(z)$ form the conclusion.

Throughout this treatise we describe the properties of a continuous function $f(z)$ by a generalized Lipschitz condition or closely related inequality. Let α be a fixed number, $0 < \alpha \leq 1$; a function $f(z)$ defined on a set E satisfies on E a Lipschitz condition of this given order α provided

$$|f(z_1) - f(z_2)| \leq L|z_1 - z_2|^{\alpha}, \tag{1.1.5}$$

where z_1 and z_2 are arbitrary points of E, and L is a constant independent of z_1 and z_2. For $\alpha = 1$ we say simply that the function satisfies a Lipschitz condition, frequently omitting the qualifying phrase "of order unity." We denote by $f'(z)$, $f''(z)$, $f'''(z)$ the first, second, and third derivatives respectively of $f(z)$; $f^{(k)}(z)$ denotes the k-th, $k = 1, 2, \ldots, m$, derivative of $f(z)$, and $f^{(0)}(z)$ denotes $f(z)$. Let E be a closed limited set bounded by a Jordan curve C; we say that a function $f(z)$ belongs to the class $L(k, \alpha)$ on C if $f(z)$ is analytic in the interior points of E, is continuous on E, and $f^{(k)}(z)$ exists on C in the one-dimensional sense and satisfies a Lipschitz condition of order α on C; in $L(k, \alpha)$ the number α is fixed, $0 < \alpha \leq 1$, and k is an integer. We say that $f(z)$ belongs to the class

Log(k,1) on C if f(z) is analytic in the interior points of E, is continuous on E, and $f^{(k)}(z)$ exists on C in the one-dimensional sense and satisfies the condition

$$(1.1.6) \quad |f^{(k)}(z_1) - f^{(k)}(z_2)| \leq L|z_1 - z_2| \cdot |\log|z_1 - z_2||,$$

where z_1 and z_2 are arbitrary points of C, and L is a constant independent of z_1 and z_2; in Log (k,1) the number k is an integer, and the condition is assumed to be satisfied merely for $|z_1 - z_2|$ sufficiently small. This latter restriction is obviously essential, for the second member of (1.1.6) reduces to zero if $|z_1 - z_2|$ is unity. We say that f(z) belongs to the class L(k, α) on E provided the function f(z) is analytic in the interior points of E, is continuous on E, and provided $f^{(k)}(z)$, defined on C in the one-dimensional sense and interior to C in the usual way, satisfies on E a Lipschitz condition of order α ; we say that f(z) belongs to the class Log (k,1) on E provided f(z) is analytic in the interior points of E, is continuous on E, and provided $f^{(k)}(z)$ defined on C in the one-dimensional sense and interior to C in the usual way, satisfies condition (1.1.6) on E. We give in the next section a detailed discussion of the relation between one-dimensional derivatives on C and two-dimensional derivatives on $\overline{C}$, and also of the relation between inequalities (1.1.5) and (1.1.6) on C and those inequalities on E.

Let C be a Jordan arc; we say that a function f(z) belongs to the class L(k, α) on C provided f(z) is continuous on C and $f^{(k)}(z)$ exists on C in the one-dimensional sense and satisfies a Lipschitz condition of order α on C; we say that f(z) belongs to the class Log (k,1) on C provided f(z) is continuous on C and $f^{(k)}(z)$ exists on C in the one-dimensional sense and satisfies condition (1.1.6) on C.

The above classifications require no modification if C consists of a finite number of Jordan curves or arcs or both. We say, for example, that $f(z)$ belongs to the class $L(k,\alpha)$ on C provided $f(z)$ belongs to the class $L(k,\alpha)$ on each component of C.

In addition to Tchebycheff approximation we consider also approximation measured by a line integral. Let $f(z)$ be defined and integrable on a set C consisting of a finite number of rectifiable Jordan curves and suppose that for each n, $n = 0, 1, 2, \ldots$, there exists $p_n(z)$ such that

$$\int_C \Delta(z) \, | f(z) - p_n(z)|^p |dz| = \epsilon_n, \lim_{n\to\infty} \epsilon_n = 0,$$

where p is a fixed positive number and $\Delta(z)$ is a non-negative norm or weight function defined and integrable on C. Degree of convergence or approximation is defined in terms of ϵ_n precisely as in the case of Tchebycheff approximation. Such approximation is called approximation in the sense of least weighted p-th powers; the concept and terms also apply to surface integrals. Here the degree of approximation depends on p as well as on $f(z)$ and C; it also depends on $\Delta(z)$ but in our study of the problem the generality of the weight function is restricted so as not to affect materially the degree of approximation. Of course we consider in this connection both Problem α and Problem β, the former in Chapter IV, and the latter in Chapter VII.

Throughout our treatment we simply write: there exists $P_n(z)$, rather than for each n, where n includes every positive integral value from 1 or 2 depending on the conditions, there exists a polynomial $P_n(z)$ of degree n in z; sometimes it is necessary in this connection to take n large in which case it is to be understood that n ranges through every positive integral value exclusive of those values less than a fixed large integer. It is necessary to exclude the values 0 and 1 of n if the function log n appears as a factor; if n to a positive power appears in the denominator it is meaningless to include the value 0;

if in the numerator the value 0 must be excluded unless the function is a constant. Of course we are interested in convergence, and these restrictions are not serious.

§1.2. LIPSCHITZ CONDITIONS AND DERIVATIVES. The classes $L(\kappa, \alpha)$ and $Log(k,1)$ defined in §1.1 are of fundamental importance in our present study; the present section is devoted to a detailed discussion of the properties of functions of these classes.

Before discussing the case of an arbitrary Jordan region we study the situation for the unit circle $\gamma: |z| = 1$, and establish some interesting inequalities with reference to Lipschitz conditions and derivatives. Let $f(z)$ be analytic in $|z| < 1$ and continuous on $|z| \leq 1$; then on the circumference $f(e^{i\theta}) \equiv F(\theta)$ is a function of the real variable θ, where $z = re^{i\theta}$, $r \leq 1$. It is easy to show that if $f(z)$ satisfies a Lipschitz condition of order α, $0 < \alpha \leq 1$, in z on γ, then it satisfies a Lipschitz condition of that same order α in θ; conversely if $F(\theta)$ satisfies a Lipschitz condition of order α in θ then $f(z)$ satisfies the same condition in z on γ. These facts are important in the application of classical results of the real domain to approximation to functions of the complex variable.

In the theorems which follow we assume $f(z)$ analytic in $|z| < 1$ and continuous on $|z| \leq 1$.

THEOREM 1.2.1. A necessary and sufficient condition that $f(z)$ belong to the class $L(0, \alpha)$ on γ is that

(1.2.1) $$|f'(re^{i\theta})| \leq L\cdot(1-r)^{\alpha-1}, \quad r < 1,$$

where L is a constant independent of r.

Suppose $f(z)$ belongs to the class $L(0, \alpha)$ on γ; then by applying the principle of the maximum to the function

$$g(z) \equiv f(ze^{ih}) - f(z),$$