APPLIED MECHANICS: SOVIET REVIEWS

Volume 1: Stability and Analytical Mechanics

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HEMISPHERE PUBLISHING CORPORATION
A member of the Taylor & Francis Group
New York Washington Philadelphia London

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APPLIED MECHANICS: SOVIET REVIEWS. Volume 1: Stability and Analytical Mechanics

Translated selectively from Itogi nauki i tekhniki, published by VINITI, with updating by the original authors. Translated by S. V. Ponomarenko.

1 2 3 4 5 6 7 8 9 0 B C B C 8 9 8 7 6 5 4 3 2 1 0 9

This book was set in Times Roman by Edwards Brothers, Inc. The editor was Janine Ludlam. Cover design by Sharon Martin DePass.
Printing and binding by BookCrafters, Inc.

Library of Congress Cataloging-in-Publication Data

Stability and analytical mechanics.
p. cm. — (Applied mechanics. Soviet reviews : v. 1)
Translated from the Russian.

1. Stability. 2. Mechanics, Analytical. I. Series.
QA871.S76 1989 89-15544
620.1—dc20 CIP

ISBN 0-89116-718-8
ISSN 1041-2646

PREFACE

Mechanics is one of the oldest fundamental sciences and has been closely connected throughout its history with all spheres of human activity. It underlies the design and make-up of various industrial and civil engineering projects, vehicles, machines, instruments, and technological gadgets. This is why mechanics still plays a leading part in both exact natural and applied sciences.

Modern mechanics faces a dual task. On the one hand, it has to further develop its theoretical basis and acquire a more precise conception of the physical and chemical processes of body interaction, and, on the other, it has to reduce choosing optimal regimes and reliable designs of machines, mechanisms, and structures to specific mathematical problems and to finding the most effective analytical and digital methods for their solution.

The publication *Applied Mechanics: Soviet Reviews* is the English translation of selected papers dealing with various problems of mechanics, both rational and applied, written by Soviet scientists for the series *Itogi nauki i tekhniki* (Advances in science and technology), a VINITI (All-Union Institute of Scientific and Technological Information) publication since 1965. Further volumes will largely include reviews from the following VINITI series: *Rational mechanics*, *Mechanics of fluids and gases*, *Mechanics of deformable solids*, and *Complex and special sections of mechan-*

ics. Each volume will contain several related reviews, updated by their authors if written earlier.

The reviews included in the present volume have been selected from the various issues of the VINITI series *Rational Mechanics* (Obshchaya mekhanika), published between 1979 and 1983, and supplemented by the authors in 1987. The original texts were published in Russian in the following issues of *Rational Mechanics:*

- Karapetyan, A. V. and Rumyantsev, V. V. Ustoychivost' konservativnykh i dissipativnykh sistem (Stability of conservative and dissipative systems). 1983, Vol. 6, 130 pp.
- Rubanovsky, V. N. Ustoychivost' ustanovivshikhsya dvizhenii slozhnykh mekhanicheskikh sistem (Stability of steady motions of complex mechanical systems). 1982, Vol. 5, pp. 62–134.
- Kunitsyn, A. L. and Markeyev, A. P. Ustoychivost' v rezonansnykh sluchayakh (Stability in resonance cases). 1979, Vol. 4, pp. 58–139.
- Sumbatov, A. S. Integraly, lineynyye otonositel'no skorostei. Obobshcheniya teoremy Yakobi (Integrals linear with respect to velocities. Generalizations of the Jacobi theorem). 1979, Vol. 4, pp. 3–57.

All four reviews in this volume were translated from the Russian by S. V. Ponomarenko

G. K. Mikhailov
V. Z. Parton

INTRODUCTION TO PART ONE

This work is a survey of results concerning the stability of the states of equilibrium and stationary motions of conservative mechanical systems, and the influence of gyroscopic and dissipative forces on stability.

The problem of stability of steady motions (equilibria and stationary motions) is of exceptional importance. In 1644, E. Torricelli formulated conditions for the stability of the state of equilibrium of a system of bodies under the action of gravitational forces. In 1788, J.-L. Lagrange [1] formulated a theorem of the equilibrium stability of an arbitrary conservative holonomic system, rigorous proof of which was offered by G. Lejeune-Dirichlet [1].

In 1877–1884, E. J. Routh [1, 2] obtained conditions for the stability of stationary motions of conservative holonomic systems with cyclic coordinates or known first integrals. W. Thomson and P. Tait [1] formulated a number of assertions in 1879 on the influence of gyroscopic and dissipative forces on stability.

Thirteen years later, A. M. Lyapunov [2] formulated fundamental principles of the modern theory of the stability of motion: he gave a general definition of stability developed two methods of stability studies based on his theorems of stability, asymptotic stability, and instability; offered the theorems of first approximation stability and instability; developed a theory of characteristic numbers; and produced a solution to the problem of stability in the simplest critical cases. Specifically, Lyapunov was the first to raise the question of the inversion of Lagrange's theorem and offer solutions in two particular cases.

N. G. Chetaev [10] made a great contribution to the theory of the stability of motion by offering a general theorem of instability and theorems of the instability of unsteady motions, developing an effective method for setting up Lyapunov's functions, and producting inversions of Lagrange's theorem for a number of cases.

The Lyapunov—Chetaev methods were further developed by many Soviet and foreign scientists, so that today the literature on the theory of motion stability is profuse.

In view of this, it is interesting to review the results of studies concerning the stability of the states of equilibrium and stationary motions of mechanical systems (both holonomic and nonholonomic), compare these results, and reveal analogies and peculiarities of the stability of motion of nonholonomic systems as compared with holonomic ones.

The earlier reviews (many of them rather detailed) dealt exclusively either with holonomic systems (N. D. Moiseyev [1], V. V. Rumyantsev [6, 8], L. Salvadori [4], P. Hagedorn [2], N. Rouche, P. Habets, and M. Laloy [1]) or nonholonomic systems (Yu. I. Neimark and N. A. Fufaev [4], A. T. Grigor'yan, B. N. Fradlin, and L. D. Roshchupkin [1], V. V. Rumyantsev and A. V. Karapetyan [1]). The only exception is a rather brief review made by S. Pluchino [1], dealing mostly with Routh's theorem for systems with known first integrals.

Besides, the aforesaid reviews do not include a number of important results received recently (this pertains both to the problem of the inversion of Lagrange's theorem and the theory of the stability of motion of nonholonomic systems).

The present review (Part One) consists of five chapters. Chapters 1 and 2 deal with the results of studying the stability of the states of equilibrium and stationary motions of holonomic systems, whereas Chapters 3 and 4 are devoted to nonholonomic systems. Simple mechanical examples are given in these chapters to illustrate general results.

Chapter 5 discusses the results of studying the states of equilibrium and stationary motions of a heavy rigid body on a horizontal plane for three types of the body's interaction with the plane, viz., an absolutely smooth plane (a conservative holonomic system), an absolutely rough plane (a conservative nonholonomic system), and a plane with friction (a dissipative holonomic system).

These results depend on the nature of the body's interaction with the horizontal plane and vividly illustrate the influence of nonholonomic relations on the form and stability of the steady motions of mechanical systems.

The Supplement includes the results of substantiating the correctness of the limiting passage from dissipative holonomic to conservative nonholonomic systems.

A. V. Karapetyan
V. V. Rumyantsev

INTRODUCTION TO PART TWO

Mechanical systems containing both subsystems with a finite number of degrees of freedom and links with distributed parameters as their components are called, for short, complex mechanical systems.

This work is a review of studies made over the past 15 years on the stability of steady motions of complex mechanical systems. It mainly concentrates on discussing the methods of analysis, rather than expounding on the results of studies dealing with the stability of motion of specific systems.

Chapter 6 of Part Two sets forth the equations of motion of complex systems, gives their first integrals, and deals with the problem of finding steady motions from the condition of stationarity of the potential energy of a given system. Chapter 7 discusses various formulations of the problems of the stability of motion of a complex system and methods for their solution. Chapters 8 and 9 examine two formulations of the problem of the stability of motion of systems containing links with distributed parameters, which have been studied earlier as applied to rigid bodies filled with liquid. One of these problem formulations (Chapter 8) is connected with the discussion of integral characteristics of the motion of continuous media and reduces the problem of the stability of motion of a system with an infinite number of degrees of freedom to the problem of stability with respect to a finite number of variables, for whose solution the method of Lyapunov functions is effective. The second problem formulation (Chapter 9) is based on the concept of the stability of the figures of "equilibrium" of a continuous medium and stability in the Lyapunov sense and reduces the problem of stability of the steady motion of a system containing links with distributed parameters to the problem of minimum of the functional of the system's changed potential energy. The solution of the problem of minimum for a rigid body with liquid was given earlier in the works of G. K. Pozharitsky [1] and G. K. Pozharitsky and V. V. Rumyantsev [1]. Some examples are also given to illustrate the application of the methods presented in Chapter 9.

Chapter 10 is a review of studies on the stability of the relative equilibrium of satellites on a circular orbit and of uniform rotations of spacecraft with elastic links.

V. N. Rubanovsky

INTRODUCTION TO PART THREE

Many problems of the stability of steady and periodic motions in analytical and celestial mechanics, and also in a number of technical sciences, lead to the study of the stability of a trivial solution of a system of common differential equations of the form

$$\frac{d\tilde{\mathbf{x}}}{dt} \equiv \dot{\tilde{\mathbf{x}}} = \mathbf{A}\tilde{\mathbf{x}} + \sum_{l=m\geq 2}^{\infty} \tilde{\mathbf{X}}^{(l)}(\tilde{\mathbf{x}},t) \tag{0.1}$$

where $\tilde{\mathbf{x}} = (x_1, \ldots, \tilde{x}_{2N})$ is a $2N$-dimensional vector; $\mathbf{A}$ is a constant $2N \times 2N$ matrix having only purely imaginary eigenvalues $\pm i\lambda_s$ $(s = 1, \ldots, N)$ and $\tilde{\mathbf{X}}^{(l)}(\tilde{\mathbf{x}}, t)$ are vector functions having as components the forms of the l-th order with respect to $\tilde{x}_1, \ldots, \tilde{x}_{2N}$ with constant or periodic in time t coefficients of the period Ω.

Such systems are of interest because, in a first approximation, they describe a certain oscillation process in which the values $|\lambda_s|$ are the frequencies of main oscillations. In the theory of stability these systems correspond to a critical case in which the problem of stability is solved via the structure of nonlinear members.

It was clear as far back as the time of A. M. Lyapunov [2] that the solution of this problem depends basically on the arithmetic properties of the values $\lambda_1, \ldots, \lambda_N$, viz., on whether they satisfy at least one of the relations

$$(\mathbf{p}, \lambda) \equiv p_1\lambda_1 + \ldots + p_N\lambda_N = \frac{2\pi}{\Omega}q, \; q = 0, \pm 1, \pm 2, \ldots \qquad (0.2)$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$; $\mathbf{p} = (p_1 \ldots, p_N)$ is a vector with integer components.

Suppose $|\mathbf{p}| = |p_1| + \ldots + |p_N|$. Then, if equality (0.2) is fulfilled, in system (0.1) an inner resonance of order $|p|$ takes place (for an autonomous system, $q = 0$).

Until recently, the above problem has been studied for the nonresonance case. The first results were obtained for the 4th-order system ($N = 2$) by G. V. Kamenkov [1] and then (by using a different method) by I. G. Malkin [1]. Subsequently, A. M. Molchanov [1], V. G. Veretennikov [1], and L. Salvadori [1] made virtually exhaustive studies of the nonresonance case. It turned out that none of the methods developed for general systems was applicable to the Hamiltonian system, basic for classical mechanics. This is why special methods (A. N. Kolmogorov [1], V. I. Arnol'd [2], and J. Moser [7]) have been developed quite recently. They are known as the KAM theories.

The resonance case has long remained outside the sphere of interest of specialists. The reason seems to be dual: the resonance problem is exceptional and very complicated. In recent years, however, interest in it has grown markedly, due to the need to further advance theoretical studies and solve strictly applied problems. Many studies have appeared recently describing quite a number of interesting properties of resonance systems, which are rather important for the general theory of the critical case under consideration.

A systematic analysis of studies concerning the theory and application of the resonance problems of stability is the purpose of this review.

It should be noted that the term "resonance" is "overloaded" today and the number of studies dealing with various resonance problems is extremely high. For this reason, the review includes only such works where the problem of stability of steady and periodic motions at inner resonance is solved rigorously, as distinguished from studies based on various modifications of asymptotic methods. Also left out here are works (interesting as they are) dealing with the studies of nonlinear resonance oscillations, the pumping of energy in the case of interacting oscillators, etc.

We do not seek to follow the chronology of various works analyzed in this review. Rather, we wish primarily to describe the most general and important formulations of the problem, methods of its solution, and results. This approach has made us divide Part Three into two parts: Chapter 11 deals only with the Hamiltonian systems, and Chapter 12 discusses the studies of systems of a general form. It is safe to say that the canonical systems have been studied fully, whereas many problems pertaining to systems of a general form still remain unsolved. Besides, the studies of the latter systems have shown that they possess a number of specific and unexpected properties impossible, in principle, in the Hamiltonian systems.

A. L. Kunitsyn
A. P. Markeyev

INTRODUCTION TO PART FOUR

It is known that the integrals of the equations of motion of mechanical systems are, as a rule, either linear or quadratic with respect to velocities. Chapter 13 of Part Four is an attempt to put together and systematize the various forms of conditions for the existence of linear integrals of equations in classical (nonrelativist) mechanics. Conditions have been established in a form that satisfies the following two requirements. First, they must be verifiable. Analytically, this means that they must be reduced to a sequence of a finite number of differentiations, integrations of full differentials, and elementary operations with the given base values (the coefficients of a system's kinetic energy, equations of constraints, forces). Second, these conditions must be invariant, i.e., they must not depend on the choice of coordinates. Indeed, a good choice of coordinates in a mechanical system, which often predetermines a successful integration of equations, is rather fortuitous. Yet the presence of linear integrals is a system's inner property, which does not fully depend on the method of its description. Note that the term "invariance" used hereafter must be taken as invariance with regard to possible point transformations of coordinates. This interpretation of invariance is, of course, very narrow, but one has to reckon with the fact that under more general transformations of variables the linearity of integrals is, generally speaking, not preserved. As was shown by L. P. Eisenhart [2], and O. Veblen and J. M. Thomas [1], the problem of determining the necessary and sufficient conditions for the existence of linear integrals in a form satisfying the above requirements is solvable in principle. As is traditional with analytical mechanics, the material is presented in generalized, Lagrangian coordinates. Transferring the results to linear nonholonomic coordinates would be formal, the more so as all the major results have been obtained in Lagrangian coordinates.

Chapter 14 deals with certain studies involving the transformation of the equations of motion of nonholonomic systems to a form of the Lagrange and Hamilton equations. The aim of this transformation is to apply the methods of analytical mechanics to holonomic systems, such as the Jacobi method, to nonholonomic systems. The survey does not deal with the generalizations of Jacobi's theorem in nonholonomic coordinates because such generalizations are mostly formal in character and have proof errors. In general, the importance of nonholonomic coordinates in the mechanics of systems with nonintegrable constraints has not been adequately evaluated. This does not pertain, however, to holonomic systems. Suffice it to say that the Hamilton equations are, essentially, equations in nonholonomic coordinates (P. V. Voronets [1]).

The repeated publication of ideas and results obtained earlier by other authors occurs rather frequently, unfortunately, in the literature on analytical mechanics. When compiling the present survey we sought to refer to original works, though we might not have always been successful. References to works containing repetitions are minimal. Some results, such as in §§4, 5, and 9, seem to be quite new and are published here for the first time.

I take this opportunity to express my sincere gratitude to V. V. Rumyantsev for his valuable counsel and attention.

A. S. Šumbatov

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