Lecture Notes in Physics

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P. Pascual

R. Tarrach

QCD: Renormalization for the Practitioner

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Authors

P. Pascual
R. Tarrach
Departamento de Fisica Teórica, Universidad de Barcelona
Diagonal 647, Barcelona 28, Spain

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PREFACE

These notes correspond to a GIFT (Grupo Interuniversitario de Física Teórica) course which was given by us in autumn 1983 at the University of Barcelona. Their main subject is renormalization in perturbative QCD and only the last chapter goes beyond perturbation theory. They are essentially self contained and their aim is to teach the student the techniques of perturbative QCD and the QCD sum rules. Their scope however is limited. A much larger coverage of QCD is given by a recent book by Yndurain [YN 83]. We both started to learn QCD from Eduardo de Rafael's notes [RA 78]; its influence is conspicuous but the blunders are ours.

We thank Pilar Udina for the typing.

Barcelona, January 1984

P. Pascual

R. Tarrach

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I. THE Q.C.D. LAGRANGIAN

Since the establishment of Q.E.D., in the late 40's, as the field theory for describing the electromagnetic interactions of charged leptons with the electromagnetic field much effort has been devoted to find a theory for the strong interactions. In the early 70's appeared quantum chromodynamics (Q.C.D.) as the theory of the strong interactions through a combined effort of many people [GE 72], [FG 72], [FG 73], [GW 73], [PO 73], [WE 73], [WE 73a] .

Q.C.D. is a renormalizable quantum field theory of the strong interactions. Its fundamental fields are Dirac spinor fields describing particles of spin ½ , called quarks, with fractionary electric charge and gauge fields corresponding to chargeless and massless particles of spin 1 , called gluons, which interact with the quarks and among themselves.

Let us denote by $q_\alpha^A(x)$ the quark fields, where the index $A = 1,2,\dots, N_f$ refers to flavor and corresponds to the observed degrees of freedom of existing hadrons such as isotopic spin, strangeness, charm, bottom, etc.. At present 5 flavors are known and usually the corresponding quarks are called: $q^1 = u$ (up) , $q^2 = d$ (down), $q^3 = s$ (strange), $q^4 = c$ (charm) and $q^5 = b$ (bottom). The index $\alpha = 1,2,\dots, N$ refers to the color degrees of freedom. The experimental evidence implies that there are three colors and the usual notation is : $q_1{}^A$ = red, $q_2{}^A$ = blue and $q_3{}^A$ = green. As it is usual for the spin ½ fields the Lagrangian density for massless free quarks can be written as

$$\mathcal{L}_0(x) = \frac{i}{2}\,\bar{q}_\alpha^A(x)\,\gamma^\mu\,\partial_\mu\, q_\alpha^A(x) - \frac{i}{2}\,[\partial_\mu\,\bar{q}_\alpha^A(x)]\,\gamma^\mu\, q_\alpha^A(x) \tag{I.1}$$

where a summation on A and α must be understood. We are using the metric $g^{\mu\nu} \equiv (1,-1,-1,-1)$. When no error is possible we will omit the flavor and/or the color indices. We will assume that SU(N) is the color group and the quark fields transform as its fundamental representation. The above given Lagrangian density is clearly invariant under global gauge color transformation. Let us now consider the local gauge color transformation

$$q_\alpha^A(x) \longrightarrow q_\alpha^{A'}(x) = G_{\alpha\beta}(x)\, q_\beta^A(x) \equiv \left[e^{-ig\, T_a \Theta_a(x)} \right]_{\alpha\beta} q_\beta^A(x) \qquad (I.2)$$

where $\theta_a(x)$ are real space-time functions, g is a real dimensionless coupling constant and T_a are the generators of SU(N) in its fundamental representation (Appendix A). Under (2) the Lagrangian density $\mathcal{L}_0(x)$ transforms as

$$\mathcal{L}_0(x) \longrightarrow \mathcal{L}_0'(x) = \mathcal{L}_0(x) + \frac{i}{2}\, \bar{q}_\alpha^A(x)\, \gamma_\mu \left[G^+(x)\, \partial^\mu G(x) \right]_{\alpha\beta} q_\beta^A(x)$$

$$- \frac{i}{2}\, \bar{q}_\alpha^A(x)\, \gamma_\mu \left[\left[\partial^\mu G^+(x) \right] G(x) \right]_{\alpha\beta} q_\beta^A(x) \qquad (I.3)$$

In order to obtain a Lagrangian density invariant under local gauge transformations we must substitute in (1) [YM 54] the usual derivative ∂^μ by a covariant one

$$\delta_{\alpha\beta}\, \partial^\mu \longrightarrow D^\mu_{\alpha\beta} \equiv \delta_{\alpha\beta}\, \partial^\mu - ig\, T^a_{\alpha\beta}\, B^\mu_a(x) \qquad (I.4)$$

where $B_a^\mu(x)$ are the (N^2-1) so-called gluon fields. The color indices will be written indistinctly as upper or lower indices. The new Lagrangian density will be invariant under local gauge transformations if $q_\alpha^A(x)$ and $D^\mu_{\alpha\beta}\ q_\beta^A(x)$ transform under (2) in the same way, i.e.

$$D^\mu_{\alpha\beta}\ q^A_\beta(x) \longrightarrow D'^\mu_{\alpha\beta}\ q^{A'}_\beta(x) = G(x)_{\alpha\gamma}\ D^\mu_{\gamma\delta}\ q^A_\delta(x) \tag{I.5}$$

where $D'^\mu_{\alpha\beta}$ denotes (4) with $B^\mu_a(x)$ substituted by the transformed field $B'^\mu_a(x)$. From the last equation we can immediately obtain that the transformed gluon fields can be written in terms of the original ones as

$$\begin{aligned} T^a_{\alpha\delta}\ B'^\mu_a(x) &= G_{\alpha\beta}(x)\ T^a_{\beta\gamma}\ B^\mu_a(x)\ G^*_{\delta\gamma}(x) \\ &\quad - \frac{i}{g}\left[\partial^\mu G_{\alpha\gamma}(x)\right] G^*_{\delta\gamma}(x) \end{aligned} \tag{I.6}$$

From (2) and (4) the transformation laws under infinitesimal local gauge transformations turn out to be

$$\begin{aligned} q^A_\alpha(x) &\longrightarrow q'^A_\alpha(x) = q^A_\alpha(x) - i g\ T^a_{\alpha\beta}\ \delta\Theta_a(x)\ q^A_\beta(x) \\ B^\mu_a(x) &\longrightarrow B'^\mu_a(x) = B^\mu_a(x) + g\ f_{abc}\ \delta\Theta_b(x)\ B^\mu_c(x) - \partial^\mu\delta\Theta_a(x) \end{aligned} \tag{I.7}$$

where $\delta\Theta_a(x)$ are the infinitesimal functions characterizing the transformation. By this procedure we have constructed a lagrangian density which is invariant under local gauge transformations:

$$\begin{aligned} \mathcal{L}(x) &= \frac{i}{2}\ \bar{q}^A_\alpha(x)\ \gamma_\mu\ D^\mu_{\alpha\beta}\ q^A_\beta(x) - \frac{i}{2}\left[D^{\mu *}_{\beta\alpha}\ \bar{q}^A_\alpha(x)\right]\gamma_\mu\ q^A_\beta(x) = \\ &= \frac{i}{2}\ \bar{q}^A_\alpha(x)\ \gamma_\mu\ \partial^\mu\ q^A_\alpha(x) - \frac{i}{2}\left[\partial^\mu\ \bar{q}^A_\alpha(x)\right]\gamma_\mu\ q^A_\alpha(x) \\ &\quad + \frac{1}{2}\ g\ \bar{q}^A_\alpha(x)\ \lambda^a_{\alpha\beta}\ \gamma_\mu\ q^A_\beta(x)\ B^\mu_a(x) \end{aligned} \tag{I.8}$$

which describes the free massless quark fields as well as their interaction with the gluon fields with a universal,real and dimensionless coupling constant g .

Sometimes it is useful to introduce the more compact notation

$$
\begin{aligned}
B^{\mu}(x) &\equiv i g \, T_a \, B_a^{\mu}(x) \\
D^{\mu} &\equiv I \partial^{\mu} - B^{\mu}(x)
\end{aligned}
\tag{I.9}
$$

where now $B^{\mu}(x)$ and D^{μ} are NxN matrices and I is the corresponding unit matrix. Using this notation the above given transformation laws are

$$
\begin{aligned}
q^A(x) &\longrightarrow q'^A(x) = G(x)\, q^A(x) \\
D^{\mu} q^A(x) &\longrightarrow D'^{\mu} q'^A(x) = G(x)\, D^{\mu} q^A(x) \\
D^{\mu} &\longrightarrow D'^{\mu} = G(x)\, D_{\mu}\, G^{-1}(x) \\
B^{\mu}(x) &\longrightarrow B'^{\mu}(x) = G(x)\, B^{\mu}(x)\, G^{-1}(x) + [\partial^{\mu} G(x)]\, G^{-1}(x)
\end{aligned}
\tag{I.10}
$$

where $q^A(x)$ is a column matrix with elements $q^A_{\alpha}(x)$ and $G(x)$ a NxN matrix with elements $G_{\alpha\beta}(x)$.

The Lagrangian density (8) does not fix the equations of motion of the gluonic fields and therefore, without destroying the local gauge invariance, we must add to it some terms in order to complete our theory. Up to this end let us define the antisymmetric field strength tensor $F^{\mu\nu}(x)$

$$F^{\mu\nu}(x) \equiv -[D^\mu, D^\nu] =$$

$$= \partial^\mu B^\nu(x) - \partial^\nu B^\mu(x) - [B^\mu(x), B^\nu(x)] \qquad (I.11)$$

which satisfies the Bianchi identity

$$[D^\rho, F^{\mu\nu}] + [D^\mu, F^{\nu\rho}] + [D^\nu, F^{\rho\mu}] = 0 \qquad (I.12)$$

The components $F_a^{\mu\nu}(x)$ are defined by

$$F^{\mu\nu}(x) \equiv i g T_a F_a^{\mu\nu}(x)$$

$$F_a^{\mu\nu}(x) = \partial^\mu B_a^\nu(x) - \partial^\nu B_a^\mu(x) + g f_{abc} B_b^\mu(x) B_c^\nu(x) \qquad (I.13)$$

where the last term reflects the non abelian character of SU(N). Taking into account the transformation law of D^μ given in (10) and the definition (11) we get immediately

$$F^{\mu\nu}(x) \longrightarrow F'^{\mu\nu}(x) = G(x) F^{\mu\nu}(x) G^{-1}(x) \qquad (I.14)$$

and hence

$$T_r [F^{\mu\nu}(x) F_{\mu\nu}(x)] = -\frac{g^2}{2} F_a^{\mu\nu}(x) F_{\mu\nu}^a(x) \qquad (I.15)$$

is a scalar under Poincaré transformations and furthermore it is invariant under local gauge transformations. We can add to (8) a term proportional to (15) and in this way we get the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2g^2} \, Tr \left[F^{\mu\nu}(x) \, F_{\mu\nu}(x) \right] + \frac{i}{2} \bar{q}^A(x) \gamma_\mu D^\mu q^A(x)$$

$$- \frac{i}{2} \overline{\left[D^\mu q^A(x) \right]} \gamma_\mu q^A(x) \tag{I.16}$$

which describes the kinetic terms for massless quarks and massless gluons as well as the interactions of quarks and gluons and the gluon selfinteractions,all of them characterized by a dimensionless coupling constant g .

As it is well known [YM 54] it is impossible to give mass to the gluons without breaking the local gauge invariance, but mass terms can be added for the quark fields without destroying this invariance. Let us now consider the most general way to give masses to the quark fields without breaking local gauge invariance. Up to this end let us introduce the left and righthanded quark fields defined as

$$q_{\alpha L,R}(x) \equiv \frac{1}{2} \left(I \pm \gamma_5 \right) q_\alpha(x) \tag{I.17}$$

where $q_\alpha(x)$ is a column matrix with N_f rows and $\gamma_5 \equiv -i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. The most general mass term with the desired properties that can be added to the Lagrangian density is

$$\mathcal{L}_M(x) = q^+_{\alpha L}(x) \, \gamma^0 M \, q_{\alpha R}(x) + q^+_{\alpha R}(x) \, \gamma^0 M^+ q_{\alpha L}(x) \tag{I.18}$$

where M is an arbitrary $N_f \times N_f$ matrix. If $\det M \neq 0$, we can write in a unique way $M = M_H U$ where M_H is the square root of MM^+ and therefore is an hermitian positive defined matrix, while $U = M_H^{-1} M$ is unitary. Then (18) can be written as

$$\mathcal{L}_M(x) = q^{+}_{\alpha L}(x)\,\gamma^0 M_H\, q'_{\alpha R}(x) + q'^{+}_{\alpha R}(x)\,\gamma^0 M_H\, q_{\alpha L}(x)$$

where $q'_{\alpha R}(x) \equiv U\, q_{\alpha R}(x)$. Furthermore if $q''_{\alpha}(x) = q'_{\alpha R}(x) + q_{\alpha L}(x)$

$$\mathcal{L}_M(x) = q^{+}_{\alpha L}(x)\,\gamma^0 M_H\,[q''_{\alpha}(x) - q_{\alpha L}(x)] + q'^{+}_{\alpha R}(x)\,\gamma^0 M_H\,[q''_{\alpha}(x) - q'_{\alpha R}(x)] =$$

$$= q^{+}_{\alpha L}(x)\,\gamma^0 M_H\, q''_{\alpha}(x) + q'^{+}_{\alpha R}(x)\,\gamma^0 M_H\, q''_{\alpha}(x) =$$

$$= \bar{q}''_{\alpha}(x)\; M_H\; q''_{\alpha}(x)$$

Since $M_H^{+} = M_H$, we can diagonalize this matrix using a unitary matrix V

$$V M_H V^{+} = M_D = \begin{vmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_{N_F} \end{vmatrix}$$

and introducing $q'''_{\alpha}(x) \equiv V\, q''_{\alpha}(x)$ we can write the mass term as

$$\mathcal{L}_M(x) = \bar{q}'''_{\alpha}(x)\; M_D\; q'''_{\alpha}(x) \qquad \text{(I.19)}$$

If $\det M = 0$ we can still write $M = M_H U$ where M_H is the one given above. The diagonalization of M_H still determines the real non-negative values of the diagonal matrix M_D giving the masses. Nevertheless, the matrices U and V are not uniquely determined by this method, unless their unitarity is explicitly imposed.

Let us check that all terms of (16) are invariant under $q_{\alpha}(x) \to q'''_{\alpha}(x)$. The most general form of the terms of (16) involving quark fields is

$$\bar{q}_\alpha(x)\,\gamma^\mu\,A_{\alpha\beta}\,q_\beta(x)$$

Let us prove its invariance

$$\bar{q}_\alpha(x)\,\gamma^\mu A_{\alpha\beta}\,q_\beta(x) = [q^+_{\alpha L}(x) + q^+_{\alpha R}(x)]\,\gamma^0\gamma^\mu A_{\alpha\beta}\,[q_{\beta L}(x) + q_{\beta R}(x)] =$$

$$= q^+_{\alpha L}(x)\,\gamma^0\gamma^\mu A_{\alpha\beta}\,q_{\beta L}(x) + q^+_{\alpha R}(x)\,\gamma^0\gamma^\mu A_{\alpha\beta}\,q_{\beta R}(x) =$$

$$= q^+_{\alpha L}(x)\,\gamma^0\gamma^\mu A_{\alpha\beta}\,q_{\beta L}(x) + q'^+_{\alpha R}(x)\,\gamma^0\gamma^\mu A_{\alpha\beta}\,q'_{\beta R}(x) =$$

$$= q^+_{\alpha L}(x)\,\gamma^0\gamma^\mu A_{\alpha\beta}\,[q''_\beta(x) - q'_{\beta R}(x)] + q'^+_{\alpha R}(x)\,\gamma^0\gamma^\mu A_{\alpha\beta}\,[q''_\beta(x) - q_{\beta L}(x)] =$$

$$= q^+_{\alpha L}(x)\,\gamma^0\gamma^\mu A_{\alpha\beta}\,q''_\beta(x) + q'^+_{\alpha R}(x)\,\gamma^0\gamma^\mu A_{\alpha\beta}\,q''_\beta(x) =$$

$$\doteq \bar{q}''_\alpha(x)\,\gamma^\mu A_{\alpha\beta}\,q''_\beta(x) = \bar{q}'''_\alpha(x)\,\gamma^\mu A_{\alpha\beta}\,q'''_\beta(x)$$

Then we can forget about the primes and the desired Lagrangian density can be written as

$$\mathcal{L}(x) = \frac{1}{2g^2}\,T_r\,[F^{\mu\nu}(x)\,F_{\mu\nu}(x)] + \frac{i}{2}\,\bar{q}^A(x)\,\gamma_\mu D^\mu q^A(x)$$

$$-\frac{i}{2}\,[\overline{D^\mu q^A(x)}]\,\gamma_\mu\,q^A(x) - m_A\,\bar{q}^A(x)\,q^A(x) \qquad \text{(I.20)}$$

or explicitly

$$\mathcal{L}(x) = -\frac{1}{4}\,[\partial_\mu B^a_\nu(x) - \partial_\nu B^a_\mu(x)]\,[\partial^\mu B^\nu_a(x) - \partial^\nu B^\mu_a(x)]$$

$$+ \frac{i}{2} \bar{q}^A_\alpha(x) \gamma^\mu \partial_\mu q^A_\alpha(x) - \frac{i}{2} [\partial_\mu \bar{q}^A_\alpha(x)] \gamma^\mu q^A_\alpha(x) - m_A \bar{q}^A_\alpha(x) q^A_\alpha(x)$$

$$+ \frac{1}{2} g \bar{q}^A_\alpha(x) \lambda^a_{\alpha\beta} \gamma_\mu q^A_\beta(x) B^\mu_a(x)$$

$$- \frac{1}{2} g f_{abc} [\partial_\mu B^a_\nu(x) - \partial_\nu B^a_\mu(x)] B^\mu_b(x) B^\nu_c(x)$$

$$- \frac{1}{4} g^2 f_{abc} f_{ade} B^b_\mu(x) B^c_\nu(x) B^\mu_d(x) B^\nu_e(x) \tag{I.21}$$

This is the Lagrangian of classical chromodynamics. The first term represents the kinetic term for the massless gluon fields; the next three terms correspond to the kinetic terms of the quark fields with the possibility of a different mass for each flavor; the fifth term describes the interaction of the quarks with the gluons and the last two terms are the self-interactions of the gluon fields due to the non-abelian character of SU(N) . The equations of motion are, in compact notation,

$$[i \gamma_\mu D^\mu - m_A] q^A(x) = 0$$

$$[D^\mu, F_{\mu\nu}(x)] = -i g^2 T_a \sum_A \bar{q}^A(x) T_a \gamma_\nu q^A(x) \tag{I.22}$$

We would like to consider before going on the global symmetries of our Lagrangian density

i) $U_B(1)$

The Lagrangian density (21) is invariant with respect to the set of one parameter transformations

$$q(x) \longrightarrow q'(x) = e^{-i\theta I} q(x) \tag{I.23}$$

where θ is a real constant and I is the unit matrix in the color and flavor spaces. To this global gauge transformation there is associated, via Noether's theorem, a baryonic current

$$J^{\mu}(x) = \frac{1}{N} \bar{q}^{A}_{\alpha}(x)\, \gamma^{\mu}\, q^{A}_{\alpha}(x) \tag{I.24}$$

where summation over color and flavor indices must be understood. This is a conserved gauge invariant current

$$\partial_{\mu} J^{\mu}(x) = 0 \tag{I.25}$$

and the associated charge

$$B \equiv \int d^3x \;\; J^{0}(t, \vec{x}) \tag{I.26}$$

is the baryonic charge, generator of the $U_B(1)$ group, which is a constant of motion.

ii) $U_1(1) \otimes U_2(1) \otimes \ldots\ldots \otimes U_{N_f}(1)$

Our Lagrangian density (21) is invariant with respect to each set of uniparametric transformations

$$q^{A}(x) \longrightarrow e^{-i\theta_A I}\, q^{A}(x) \quad , \quad A = 1, 2, \ldots, N_f \tag{I.27}$$

where θ_A are real constants and I is the unit matrix in the color space. To each flavor A there is associated a global symmetry $U_A(1)$ and therefore our Lagrangian is invariant under the group $U_1(1) \otimes U_2(1) \otimes \ldots \otimes U_{N_f}(1)$. The associated gauge invariant currents are

$$J^{A}_{\mu}(x) = \bar{q}^{A}_{\alpha}(x)\, \gamma_{\mu}\, q^{A}_{\alpha}(x) \quad , \quad A = 1, 2, \ldots, N_f \tag{I.28}$$

where a summation over color indices must be understood. These currents are conserved and the corresponding charges are the generators of the group. These symmetries correspond to the separate conservation of each flavor in the strong interactions.

Notice furthermore that if $m_i = m_j$ then $\mathcal{L}(x)$ has a global symmetry larger than $U_i(1) \otimes U_j(1)$; it is invariant under the group of transformations SU(2) acting on the space $(q^i(x)\ ,\ q^j(x))$. If all masses are equal the global symmetry group is $SU(N_f)$.

iii) $SU_L(N_f) \otimes SU_R(N_f)$

Let us now consider the global transformation acting only on the flavor indices

$$
\begin{aligned}
q_\alpha(x) &\longrightarrow q'_\alpha(x) = e^{-i\Theta^A T^A} q_\alpha(x) \\
q_\alpha(x) &\longrightarrow q'_\alpha(x) = e^{-i\Theta^A T^A \gamma_5} q_\alpha(x)
\end{aligned}
\tag{I.29}
$$

where Θ_A is a set of $(N_f^2 - 1)$ real constants and T^A are the generators of $SU(N_f)$ in the fundamental representation. These transformations are global symmetries of our Lagrangian density only if the mass terms are absent: $m_A = 0$. Via Noether's theorem we can associate to the transformations (29) the gauge invariant currents

$$
\begin{aligned}
V_A^\mu(x) &= \bar{q}^y_\alpha(x)\, \gamma^\mu\, (T^A)_{yz}\, q^z_\alpha(x) \\
A_A^\mu(x) &= \bar{q}^y_\alpha(x)\, \gamma^\mu \gamma_5\, (T^A)_{yz}\, q^z_\alpha(x)
\end{aligned}
\tag{I.30}
$$

which are the vector and axial vector currents of the current algebra of Gell-Mann [GE 64]. Notice

$$
\begin{aligned}
\partial_\mu V_A^\mu(x) &= i\,(m_y - m_z)\, \bar{q}^y_\alpha(x)\, (T^A)_{yz}\, q^z_\alpha(x) \\
\partial_\mu A_A^\mu(x) &= i\,(m_y + m_z)\, \bar{q}^y_\alpha(x)\, \gamma_5\, (T^A)_{yz}\, q^z_\alpha(x)
\end{aligned}
\tag{I.31}
$$

so that the currents are conserved if the quark masses are zero.

Assuming that $q^A_\alpha(x)$ is a quantum field we would like to compute the equal time commutation relations of these currents. All these currents have the general structure $X(x) \equiv q^+(x)\, O\, q(x)$ where O is a matrix acting on color, flavor and spin indices. Let us remember that

$$\delta(x^0 - y^0)\,\{ q_i(x) \,,\, q_j^+(y) \} = \delta_{ij}\; \delta^{(4)}(x-y) \tag{I.32}$$

where the subindex stands for color, flavor and spin components. Then it is immediate to prove that

$$\begin{aligned} \delta(x^0 - y^0)\,[\, q^+(x)\, O\, q(x) \,,\; q^+(y)\, O'\, q(y) \,] &= \\ &= q^+(x)\,[\, O , O' \,]\, q(x)\; \delta^{(4)}(x-y) \end{aligned} \tag{I.33}$$

In the cases that we are interested in $O = I \otimes \lambda_A/2 \otimes \Gamma$, where I is the unit matrix in the color space, $\lambda_A/2$ are the generators of the $SU(N_f)$ flavor group in its fundamental representation and Γ are matrices acting on the spin indices. It is convenient to introduce

$$\lambda_0 \equiv \sqrt{\frac{2}{N_f}}\; I$$

and then relations (A.6) and (A.12) can be written

$$\lambda_A \lambda_B = [\, d_{ABC} + i f_{ABC} \,]\, \lambda_C \qquad , \qquad \mathrm{Tr}\,[\, \lambda_A \lambda_B \,] = 2\, \delta_{AB} \tag{I.34}$$

where the indices run from zero to $N_f^2 - 1$. Then