A FIRST COURSE IN

Mathematical Modeling

THIRD EDITION

Frank R. Giordano   Maurice D. Weir   William P. Fox

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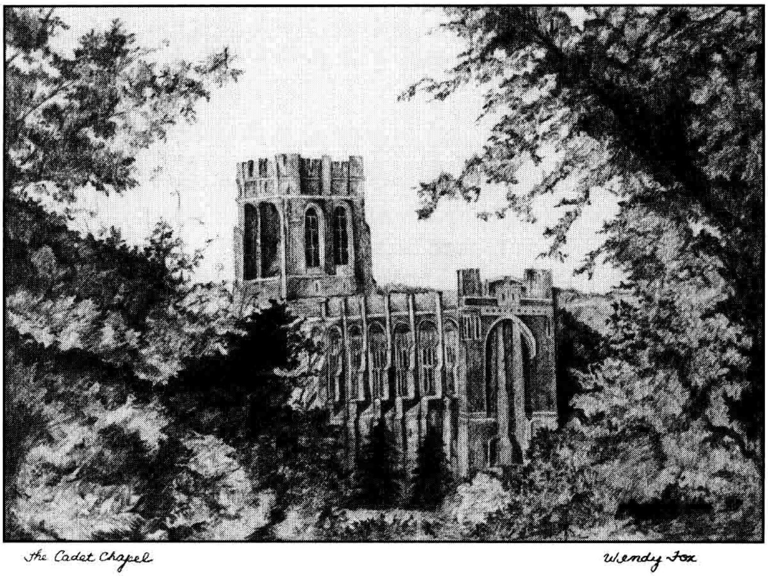
Dedicated to our mentors, colleagues, and students
who continue to motivate mathematical modeling
in our lives, our teaching, and our research,

especially to

David C. Cameron
Teacher, Scholar, Soldier, Mentor, and Friend



A teacher affects eternity.
He knows not where his influence stops.

———Adams

Preface

To facilitate an early initiation of the modeling experience, the first edition of this text was designed to be taught concurrently or immediately after an introductory business or engineering calculus course. In the second edition, we added chapters treating discrete dynamical systems, linear programming and numerical search methods, and an introduction to probabilistic modeling. Additionally, we expanded our introduction of simulation. In this edition we have included solution methods to some simple dynamical systems to reveal their long-term behavior. We have also added basic numerical solution methods to the chapters covering modeling with differential equations. The text has been reorganized into two parts: Part One, Discrete Modeling (Chapters 1–8), and Part Two, Continuous Modeling (Chapters 9–12). This organizational structure allows for teaching an entire modeling course based on Part One and which does not require the calculus. Part Two then addresses continuous models based on optimization and differential equations which can be presented concurrently with freshman calculus. The text gives students an opportunity to cover all phases of the mathematical modeling process. The new CD-ROM accompanying the text contains software, additional modeling scenarios and projects, and a link to past problems from the Mathematical Contest in Modeling. We thank Sol Garfunkel and the COMAP staff for preparing the CD and for their support of modeling activities that we refer to under Resource Materials below.

Goals and Orientation

The course continues to be a bridge between the study of mathematics and the applications of mathematics to various fields. The course affords the student an early opportunity to see how the pieces of an applied problem fit together. The student investigates meaningful and practical problems chosen from common experiences encompassing many academic disciplines, including the mathematical sciences, operations research, engineering, and the management and life sciences.

This text provides an introduction to the entire modeling process. The student will have occasions to practice the following facets of modeling and enhance their problem-solving capabilities:

1. *Creative and Empirical Model Construction:* Given a real-world scenario, the student learns to identify a problem, make assumptions and collect data, propose a model, test the assumptions, refine the model as necessary, fit the model to data if appropriate, and analyze the underlying mathematical structure of the model to appraise the sensitivity of the conclusions when the assumptions are not precisely met.

2. *Model Analysis:* Given a model, the student learns to work backward to uncover the implicit underlying assumptions, assess critically how well those assumptions fit the scenario at hand, and estimate the sensitivity of the conclusions when the assumptions are not precisely met.

3. *Model Research:* The student investigates a specific area to gain a deeper understanding of some behavior and learns to use what has already been created or discovered.

Student Background and Course Content

Because our desire is to initiate the modeling experience as early as possible in the student's program, the only prerequisite for Chapters 9, 10, and 11 is a basic understanding of single-variable differential and integral calculus. Although some unfamiliar mathematical ideas are taught as part of the modeling process, the emphasis is on using mathematics already known by the students after completing high school. This emphasis is especially true in Part One. The modeling course will then motivate students to study the more advanced courses such as linear algebra, differential equations, optimization and linear programming, numerical analysis, probability, and statistics. The power and utility of these subjects are intimated throughout the text.

Further, the scenarios and problems in the text are not designed for the application of a particular mathematical technique. Instead, they demand thoughtful ingenuity in using fundamental concepts to find reasonable solutions to "open-ended" problems. Certain mathematical techniques (such as Monte Carlo simulation, curve fitting, and dimensional analysis) are presented because often they are not formally covered at the undergraduate level. Instructors should find great flexibility in adapting the text to meet the particular needs of students through the problem assignments and student projects. We have used this material to teach courses to both undergraduate and graduate students, and even as a basis for faculty seminars.

Organization of the Text

The organization of the text is best understood with the aid of Figure 1. The first eight chapters constitute Part One and require only precalculus mathematics as a prerequisite. We begin with the idea of modeling *change* using simple finite difference equations. This approach is quite intuitive to the student and provides us with several concrete models to support our discussion of the modeling process in Chapter 2. There we classify models, analyze the modeling process, and construct several proportionality models or submodels which are then revisited in the next two chapters. In Chapter 3 the student is presented with three criteria for fitting a specific curve-type to a collected data set, with emphasis on the least-squares criterion. Chapter 4 addresses the problem of capturing the trend of a collected set of data. In this empirical construction process, we begin with fitting simple one-term models approximating collected data sets and progress to more sophisticated interpolating models, including polynomial smoothing models and cubic splines.

Simulation models are discussed in Chapter 5. An empirical model is fit to some collected data, and then Monte Carlo simulation is used to duplicate the behavior being investigated. The presentation motivates the eventual study of probability and statistics.

Chapter 6 provides an introduction to probabilistic modeling. The topics of Markov processes, reliability, and linear regression are introduced, building on scenarios and analysis presented previously. Chapter 7 addresses the issue of finding the best-fitting model using the other two criteria presented in Chapter 3. Linear programming is the method used for finding the "best" model for one of the criteria, and numerical search techniques can be used for the other. The chapter concludes with an introduction to numerical search methods including the dichotomous and golden section methods. Part One ends with Chapter 8, which is devoted to dimensional analysis, a topic of great importance in the physical sciences and engineering.

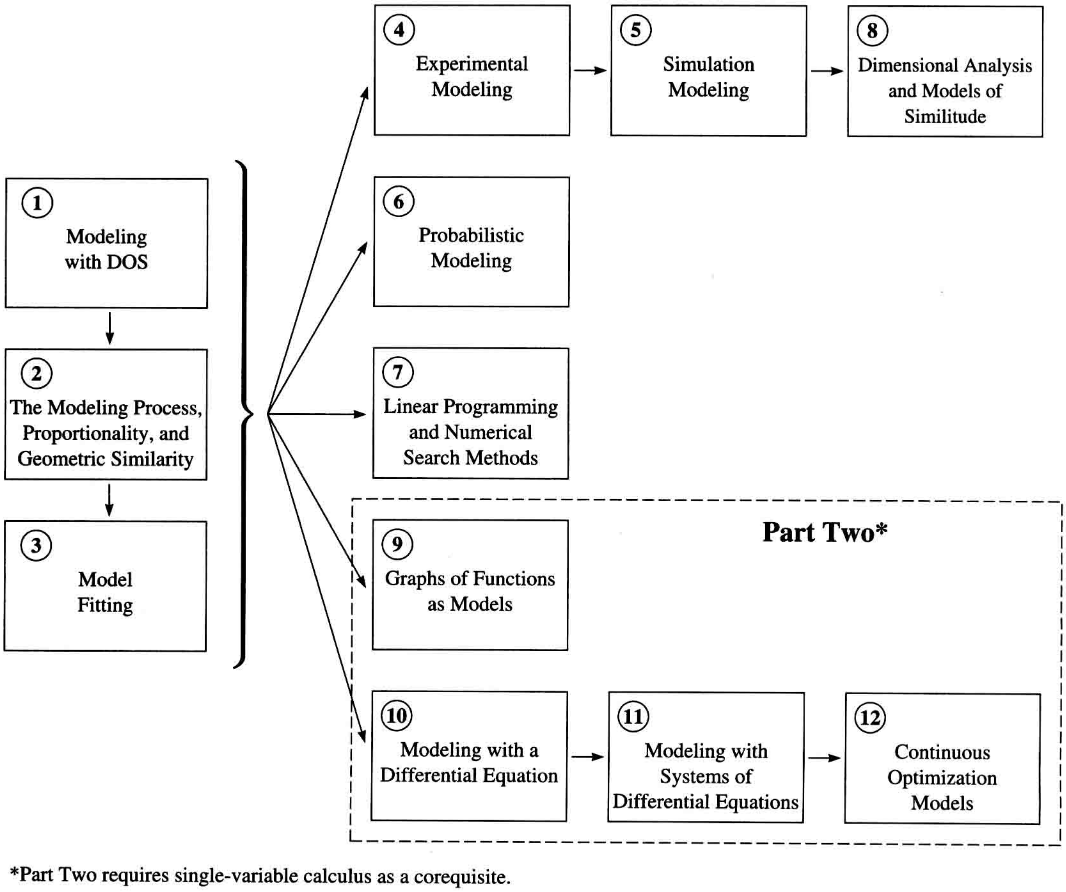


Figure 1
Chapter organization and progression

Part Two is dedicated to the study of continuous models. Chapter 9 treats the construction of continuous graphical models and explores the sensitivity of the models constructed to the assumptions underlying them. In Chapters 10 and 11 we model dynamic (time varying) scenarios. These chapters build on the discrete analysis presented in Chapter 1 by now considering situations where time is varying continuously. Chapter 12 is devoted to the study of continuous optimization. Students get the opportunity to solve continuous optimization problems requiring only the application of elementary calculus and are introduced to constrained optimization problems as well.

Student Projects

Student projects are an essential part of any modeling course. This text includes projects in creative and empirical model construction, model analysis, and model research. Thus we recommend a course consisting of a mixture of projects in all three facets of modeling. These projects are most instructive if they address scenarios that have no unique solution. Some projects should include *real* data that students are either given or can *readily* collect. A combination of individual and group projects can also be valuable. Individual projects are appropriate in those parts of the course in which the instructor wishes to emphasize the development of individual modeling skills. However, the inclusion of a group project early in the course gives students the exhilaration of a "brainstorming" session. A variety of projects is suggested in the text, such as constructing models for various scenarios, completing UMAP[1] modules, or researching a model presented as an example in the text or class. It is valuable for each student to receive a mixture of projects requiring either model construction, model analysis, or model research for variety and confidence building throughout the course. Students might also choose to develop a model in a scenario of particular interest, or analyze a model presented in another course. We recommend five to eight short projects in a typical modeling course. Detailed suggestions on how student projects can be assigned and used are included in the Instructor's Manual that accompany this text.

In terms of the number of scenarios covered throughout the course, as well as the number of homework problems and projects assigned, we have found it better to pursue a few that are developed carefully and completely. We have provided many more problems and projects than can reasonably be assigned to allow for a wide selection covering many different application areas.

The Role of Computation

Although many chapters of the text do not require computing capability,[2] computation does play an important role in any realistic modeling course. We have found

[1] UMAP modules are developed and distributed through COMAP, Inc., 57 Bedford Street, Suite 210, Lexington, MA 02173.

[2] Chapters 2, 6, and 8–12 do not require computing capability.

a combination of graphing calculators and computers to be advantageous throughout the course. The use of a spreadsheet is beneficial in Chapters 1, 5, and 7, and the capability for graphical displays of data is enormously useful, even essential, whenever data is provided. Students will find computers useful, too, in transforming data, least-squares curve fitting, divided difference tables and cubic splines, programming simulation models, linear programming and numerical search methods, and numerical solutions to differential equations. The CD accompanying this text provides some basic technology tools that students and instructors can use as a foundation for modeling with technology. Several FORTRAN executable programs are provided to execute the methodologies presented in Chapter 4. Also included is a tutorial on the computer algebra system MAPLE and its use for this text.

Resource Materials

We have found material provided by the Consortium for Mathematics and Its Application (COMAP) to be outstanding and particularly well suited to the course we propose. Individual modules for the undergraduate classroom, UMAP Modules, may be used in a variety of ways. First, they may be used as instructional material to support several lessons. In this mode a student completes the self-study module by working through its exercises (the detailed solutions provided with the module can be conveniently removed before it is issued). Another option is to put together a block of instruction using one or more UMAP modules suggested in the projects sections of the text. The modules also provide excellent sources for "model research," because they cover a wide variety of applications of mathematics in many fields. In this mode, a student is given an appropriate module to research and is asked to complete and report on the module. Finally, the modules are excellent resources for scenarios for which students can practice model construction. In this mode the instructor writes a scenario for a student project based on an application addressed in a particular module and uses the module as background material, perhaps having the student complete the module at a later date. The CD accompanying the text contains most of the UMAPS referenced throughout. Information on the availability of newly developed interdisciplinary projects can be obtained by writing COMAP at the address given previously, calling COMAP at 1-800-772-6627, or electronically: order@comap.com

A great source of student-group projects are the Mathematical Contest in Modeling (MCM) and the Interdisciplinary Mathematical Contest in Modeling (IMCM). These projects can be taken from the link provided on the CD and tailored by the instructor to meet specific goals for their class. These are also good resources to prepare teams to compete in the MCM and IMCM contests currently sponsored by the National Security Agency (NSA) and COMAP. The contest is sponsored by COMAP with funding support from the National Security Agency, the Society of Industrial and Applied Mathematics, the Institute for Operations Research and the Management Sciences, and the Mathematical Association of America. Additional information concerning the contest can be obtained by contacting COMAP, or visiting their website at www.comap.com.

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The production of any mathematics text is a complex process and we have been especially fortunate in having a superb and creative production staff at Brooks/Cole. In particular, we express our thanks to Craig Barth, our editor for the first edition, Gary Ostedt, the second edition, and Gary Ostedt and Bob Pirtle, our editors for this edition. For this edition we are especially grateful to Tom Ziolkowski, our marketing manager; Tom Novack, our production editor; Merrill Peterson and Matrix Productions for production service; and Amy Moellering for her superb copyediting and typesetting. We are especially grateful to Wendy Fox for providing her drawing of the Cadet Chapel at West Point for the dedication page.

Finally, we are grateful to our wives—Judi Giordano, Gale Weir, and Wendy Fox—for their inspiration and support.

Frank R. Giordano
Maurice D. Weir
William P. Fox

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1 Modeling Change

Introduction

To help us better understand our world, we often describe a particular phenomenon mathematically (by means of a function or an equation, for instance). Such a **mathematical model** is an idealization of the real-world phenomenon and never a completely accurate representation. Although any model has its limitations, a good one can provide valuable results and conclusions. In this chapter we direct our attention to modeling change.

Mathematical Models

In modeling our world, we are often interested in predicting the value of a variable at some time in the future. Perhaps it is a population, a real estate value, or the number of people with a communicative disease. Often a mathematical model can help us understand a behavior better or aid us in planning for the future. Let's think of a mathematical model as a mathematical construct designed to study a particular real-world system or behavior of interest. The model allows us to reach mathematical conclusions about the behavior, as illustrated in Figure 1.1. These conclusions can be interpreted to help a decision maker plan for the future.

Simplification

Most models simplify reality. Generally, models can only approximate real-world behavior. One very powerful simplifying relationship is **proportionality**.

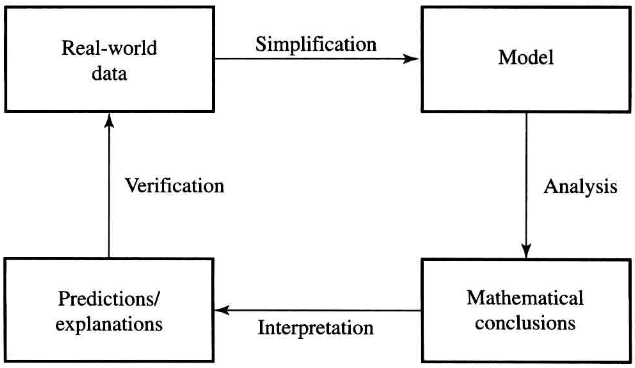

Figure 1.1
A flow of the modeling process beginning with an examination of real-world data

DEFINITION

Two variables y and x are **proportional** (to each other) if one is always a constant multiple of the other; that is, if

$$y = kx$$

for some nonzero constant k. We write $y \propto x$.

The definition means that the graph of y versus x lies along a straight line through the origin. This graphical observation is useful in testing whether a given data collection reasonably assumes a proportionality relationship. If a proportionality is reasonable, a plot of one variable against the other should approximate a straight line through the origin. Here is an example.

EXAMPLE 1 Testing for Proportionality

Table 1.1 Spring–mass system

Mass	Elong
50	1.000
100	1.875
150	2.750
200	3.250
250	4.375
300	4.875
350	5.675
400	6.500
450	7.250
500	8.000
550	8.750

Consider a spring–mass system, such as the one shown in Figure 1.2. We conduct an experiment to measure the stretch of the spring as a function of the mass (measured as weight) placed on the spring. Consider the data collected for this experiment, displayed in Table 1.1. A scatterplot graph of the stretch or elongation of the spring versus the mass or weight placed on it reveals an approximate straight line passing through the origin. (Figure 1.3).

The data appear to follow the proportionality rule that elongation e is proportional to the mass m, or symbolically, $e \propto m$. The straight line appears to pass through the origin. This geometric understanding allows us to look at the data to determine if proportionality is a reasonable simplifying assumption and, if so, to estimate the slope k. In this case, the assumption appears valid, so we estimate the constant of proportionality by picking the two points (200, 3.25) and (300, 4.875) as lying along the straight line. We calculate the slope of the line joining these points as

$$slope = \frac{4.875 - 3.25}{300 - 200} = 0.01625$$

Figure 1.2
Spring-mass system