volume 36

lecture notes in pure and applied mathematics

complex analysis

proceedings of the s.u.n.y. brockport conference

edited by
Sanford S. Miller

Complex Analysis

Proceedings of the S.U.N.Y. Brockport Conference

Edited by

Sanford S. Miller
State University of New York
Brockport, New York

MARCEL DEKKER, INC. New York and Basel

Library of Congress Cataloging in Publication Data

State University of New York Conference on Complex
Function Theory, State University College at
Brockport, 1976.
Complex analysis.

(Lecture notes in pure and applied mathematics ;
v. 36)
1. Functions of complex variables--Congresses.
2. Functions, Entire--Congresses. 3. Univalent
functions--Congresses. I. Miller, Sanford S.,
(Date) II. Title.
QA331.S72 1976 515'.9 78-2217
ISBN 0-8247-6725-X

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

Current printing (last digit):
10 9 8 7 6 5 4 3 2 1

PRINTED IN THE UNITED STATES OF AMERICA

PURE AND APPLIED MATHEMATICS

A Program of Monographs, Textbooks, and Lecture Notes

Contributions to *Lecture Notes in Pure and Applied Mathematics* are reproduced by direct photography of the author's typewritten manuscript. Potential authors are advised to submit preliminary manuscripts for review purposes. After acceptance, the author is responsible for preparing the final manuscript in camera-ready form, suitable for direct reproduction. Marcel Dekker, Inc. will furnish instructions to authors and special typing paper. Sample pages are reviewed and returned with our suggestions to assure quality control and the most attractive rendering of your manuscript. The publisher will also be happy to supervise and assist in all stages of the preparation of your camera-ready manuscript.

LECTURE NOTES
IN PURE AND APPLIED MATHEMATICS

1. *N. Jacobson,* Exceptional Lie Algebras
2. *L.-Å. Lindahl and F. Poulsen,* Thin Sets in Harmonic Analysis
3. *I. Satake,* Classification Theory of Semi-Simple Algebraic Groups
4. *F. Hirzebruch, W. D. Newmann, and S. S. Koh,* Differentiable Manifolds and Quadratic Forms
5. *I. Chavel,* Riemannian Symmetric Spaces of Rank One
6. *R. B. Burckel,* Characterization of C(X) Among Its Subalgebras
7. *B. R. McDonald, A. R. Magid, and K. C. Smith,* Ring Theory: Proceedings of the Oklahoma Conference
8. *Y.-T. Siu,* Techniques of Extension of Analytic Objects
9. *S. R. Caradus, W. E. Pfaffenberger, and B. Yood,* Calkin Algebras and Algebras of Operators on Banach Spaces
10. *E. O. Roxin, P.-T. Liu, and R. L. Sternberg,* Differential Games and Control Theory
11. *M. Orzech and C. Small,* The Brauer Group of Commutative Rings
12. *S. Thomeier,* Topology and Its Applications
13. *J. M. López and K. A. Ross,* Sidon Sets
14. *W. W. Comfort and S. Negrepontis,* Continuous Pseudometrics
15. *K. McKennon and J. M. Robertson,* Locally Convex Spaces
16. *M. Carmeli and S. Malin,* Representations of the Rotation and Lorentz Groups: An Introduction
17. *G. B. Seligman,* Rational Methods in Lie Algebras
18. *D. G. de Figueiredo,* Functional Analysis: Proceedings of the Brazilian Mathematical Society Symposium
19. *L. Cesari, R. Kannan, and J. D. Schuur,* Nonlinear Functional Analysis and Differential Equations: Proceedings of the Michigan State University Conference
20. *J. J. Schäffer,* Geometry of Spheres in Normed Spaces
21. *K. Yano and M. Kon,* Anti-Invariant Submanifolds
22. *W. V. Vasconcelos,* The Rings of Dimension Two
23. *R. E. Chandler,* Hausdorff Compactifications
24. *S. P. Franklin and B. V. S. Thomas,* Topology: Proceedings of the Memphis State University Conference
25. *S. K. Jain,* Ring Theory: Proceedings of the Ohio University Conference
26. *B. R. McDonald and R. A. Morris,* Ring Theory II: Proceedings of the Second Oklahoma Conference
27. *R. B. Mura and A. Rhemtulla,* Orderable Groups
28. *J. R. Graef,* Stability of Dynamical Systems: Theory and Applications
29. *H.-C. Wang,* Homogeneous Banach Algebras
30. *E. O. Roxin, P.-T. Liu, and R. L. Sternberg,* Differential Games and Control Theory II
31. *R. D. Porter,* Introduction to Fibre Bundles
32. *M. Altman,* Contractors and Contractor Directions Theory and Applications
33. *J. S. Golan,* Decomposition and Dimension in Module Categories
34. *G. Fairweather,* Finite Element Galerkin Methods for Differential Equations
35. *J. D. Sally,* Numbers of Generators of Ideals in Local Rings
36. *S. S. Miller,* Complex Analysis: Proceedings of the S.U.N.Y. Brockport Conference

Other Volumes in Preparation

PREFACE

The State University of New York Conference on Complex Function Theory was held on June 7, 8, and 9, 1976 at the State University College at Brockport, New York. The main emphasis of the conference was on recent developments in the fields of univalent function theory and the theory of entire functions.

The principal lecturer at the conference was Malcolm S. Robertson, former Unidel Professor of Mathematics at the University of Delaware. Professor Robertson gave two talks on complex powers of univalent functions and an application of subordination theory. There were a total of nineteen other invited lecturers who discussed their recent research in complex function theory.

This volume contains a collection of the papers presented at the conference plus a collection of unsolved problems in function theory that were proposed by the participants.

Funding for this conference was supported by a grant from the State University of New York Conversations in the Disciplines Program and from the Department of Mathematics at the State University College at Brockport. Special thanks are due K. Thomas Finley, Dean of the Division of Science and Mathematics, for his administrative help, and Marie Bell for her secretarial assistance.

Sanford S. Miller

CONTRIBUTORS TO THIS VOLUME

Roger W. Barnard
Texas Tech University, Lubbock, Texas

Louis Brickman
State University of New York at Albany, Albany, New York

Douglas M. Campbell
Brigham Young University, Provo, Utah

Paul J. Eenigenburg
Western Michigan University, Kalamazoo, Michigan

Wolfgang H. Fuchs
Cornell University, Ithaca, New York

P. M. Gauthier
Université de Montréal, Montreal, Canada

Walter Hengartner
Université Laval, Quebec, Canada

James A. Jenkins
Washington University, St. Louis, Missouri

Jack Lamoreaux
Brigham Young University, Provo, Utah

George B. Leeman, Jr.
IBM Thomas J. Watson Research Center, Yorktown Heights, New York

Z. Lewandowski
Maria Curie-Sklodowska University, Lublin, Poland

Richard J. Libera
University of Delaware, Newark, Delaware

Thomas H. MacGregor
State University of New York at Albany, Albany, New York

Sanford S. Miller
State University of New York, Brockport, New York

Edward J. Moulis, Jr.
United States Naval Academy, Annapolis, Maryland

James D. Nelson
Western Michigan University, Kalamazoo, Michigan

J. H. Neuwirth
University of Connecticut, Storrs, Connecticut

John Quine
The Florida State University, Tallahassee, Florida

Malcolm S. Robertson
Lawson Avenue, West Vancouver, British Columbia, Canada

Albert Schild
Temple University, Philadelphia, Pennsylvania

S. M. Shah
University of Kentucky, Lexington, Kentucky

Hari Shankar
Ohio University, Athens, Ohio

Herb Silverman
University of Delaware, Newark, Delaware

Jan Szynal
Université de Montréal, Montreal, Canada and
Maria Curie-Sklodowska University, Lublin, Poland

David Tepper
Clark University, Worcester, Massachusetts

S. Wajler
Maria Curie-Sklodowska University, Lublin, Poland

Joseph Warren
St. John's University, Jamaica, New York

Donald Wilken
State University of New York at Albany, Albany, New York

Eliguisz J. Złotkiewicz
Maria Curie-Sklodowska University, Lublin, Poland

LIST OF SPEAKERS

PRINCIPAL LECTURER
Malcolm S. Robertson
Former Unidel Professor of Mathematics
University of Delaware
Complex powers of univalent functions and an application of subordination theory.

LECTURERS
Roger W. Barnard
Texas Tech University
A generalization of Study's theorem on convex maps.

Louis Brickman
State University of New York at Albany
Subordination and insuperable elements.

Douglas M. Campbell
Brigham Young University
A true proof of a lemma of Schiffer on continua with trivial limit directions.

Paul J. Eenigenburg
Western Michigan University
On area and subordination.

Wolfgang H. Fuchs
Cornell University
Approximation by polynomials on disjoint sets.

Walter Hengartner
Université Laval
Simultaneous approximation and interpolation.

James A. Jenkins
Washington University
A theorem of Caratheodory-Fejer.

George B. Leeman, Jr.
IBM Thomas J. Watson Research Center
The seventh coefficient of odd symmetric functions.

Richard J. Libera
University of Delaware
Loewner's equation for some classes of bounded functions.

Thomas H. MacGregor
State University of New York at Albany
Estimates on integral means of univalent functions.

Edward J. Moulis, Jr.
United States Naval Academy
The univalence of a class of analytic functions.

John Quine
The Florida State University
Univalent polynomials.

Albert Schild
Temple University
Convolutions of univalent functions with negative coefficients.

S. M. Shah
University of Kentucky
Approximation of a class of meromophic functions by rational functions on the positive real axis, and error estimates.

Hari Shankar
Ohio University
Weighted means of analytic functions.

Herb Silverman
University of Delaware
Applications of a theorem of Zmorovič.

Jan Szynal
Université de Montreal and Maria Curie-Sklodowska University
Covering sets and majorization of functions.

David Tepper
Clark University
Covering properties of H^p functions.

Joseph Warren
St. John's University
An elementary proof of Collingwood-Cartwright's main theorem in the small.

CONFERENCE PARTICIPANTS

Professor Roger W. Barnard
Texas Tech University
Lubbock, Texas

Professor Louis Brickman
State University of New York at Albany
Albany, New York

Professor Douglas M. Campbell
Brigham Young University
Provo, Utah

Professor Bettye Ann Case
Tallahassee Community College
Tallahassee, Florida

Professor Paul J. Eenigenburg
Western Michigan University
Kalamazoo, Michigan

Thomas Falconieri
State University of New York at Albany
Albany, New York

Jinfu Feng
State University of New York at Albany
Albany, New York

Professor Wolfgang H. Fuchs
Cornell University
Ithaca, New York

Professor James Guyker
State University of New York
College at Buffalo
Buffalo, New York

Professor David J. Hallenbeck
University of Delaware
Newark, Delaware

Professor Walter Hengartner
Université Laval
Quebec, Canada

Mark Hollingsworth
University of Cincinnati
Cincinnati, Ohio

Professor James A. Jenkins
Washington University
St. Louis, Missouri

Dr. William E. Kirwan
Program Director, Classical Analysis
Mathematical Sciences Section
National Science Foundation
Washington, D.C.

Dr. George B. Leeman, Jr.
IBM Thomas J. Watson Research Center
Yorktown Heights, New York

Professor Richard J. Libera
University of Delaware
Newark, Delaware

Professor Albert E. Livingston
University of Delaware
Newark, Delaware

Professor Thomas H. MacGregor
State University of New York at Albany
Albany, New York

Professor Louis Mahoney
State University of New York at Binghamton
Binghamton, New York

Professor Edward P. Merkes
University of Cincinnati
Cincinnati, Ohio

Professor John Milcetich
The Federal City College
Washington, D.C.

Professor James Miller
West Virginia University
Morgantown, West Virginia

Professor Sanford S. Miller
State University of New York
Brockport, New York

Professor David Minda
University of Cincinnati
Cincinnati, Ohio

Professor Edward J. Moulis, Jr.
United States Naval Academy
Annapolis, Maryland

Yusaf Aba Muhanna
State University of New York at Albany
Albany, New York

Professor Kazumi Nakano
State University of New York
Brockport, New York

Professor James W. Noonan
Holy Cross College
Worcester, Massachusetts

Professor Art Obrock
Case Western Reserve University
Cleveland, Ohio

Professor Ko Choul Oum
State University of New York
College at Buffalo
Buffalo, New York

Kent Pearce
State University of New York at Albany
Albany, New York

Professor John Quine
The Florida State University
Tallahassee, Florida

Professor Malcolm S. Robertson
Lawson Avenue
West Vancouver
British Columbia, Canada

Professor Theron Rockhill
State University of New York
Brockport, New York

Professor Albert Schild
Temple University
Philadelphia, Pennsylvania

Professor Walter Schneider
Carleton University
Ottawa, Ontario, Canada

Professor Dorothy B. Shaffer
Fairfield University
Fairfield, Connecticut

Professor S. M. Shah
University of Kentucky
Lexington, Kentucky

Professor Hari Shankar
Ohio University
Athens, Ohio

Professor Herb Silverman
University of Delaware
Newark, Delaware

Professor Evelyn M. Silvia
University of California
Davis, California

Professor Linda R. Sons
Northern Illinois University
Dekalb, Illinois

Professor Jan Szynal
Université de Montreal
Montreal, Canada and
Maria Curie-Sklodowska University
Lublin, Poland

Donald N. Telage
University of Delaware
Newark, Delaware

Professor David Tepper
Clark University
Worcester, Massachusetts

Joseph Wald
University of Delaware
Newark, Delaware

Professor Joseph Warren
St. John's University
Jamaica, New York

Professor Donald Wright
University of Cincinnati
Cincinnati, Ohio

Steve Zemyan
University of Maryland
College Park, Maryland

Professor Michael Ziegler
Marquette University
Milwaukee, Wisconsin

CONTENTS

COMPLEX POWERS OF p-VALENT FUNCTIONS AND SUBORDINATION

Malcolm S. Robertson

Let $F(z)$ be regular, univalent and non-vanishing in $D\ \{z: |z| < 1\}$ with the normalization $F(0) = 1$. Let X be a set of complex numbers x. By bounding the argument of $F(z)$ a suitable set X is obtained so that $f(z) = [F(z)]^x$ is univalent in D. The order $p = p(x)$ of the circumferentially mean p-valent function f is obtained. Properties of the functions $f(z)$ such as rate of growth, coefficient estimates are studied. Applications are made to functions f when $F(z) = \left[\frac{s(z)}{K(z)}\right]^{\frac{1}{2}}$, where $s(z)$ is starlike and $K(z)$ is the Koebe function $z(1-z)^{-2}$. Application is also made to obtain new results for spirallike functions. The theory of subordination and quasi-subordination is used extensively.

AMS(MOS) subject classification (1970). Primary 30A34.

Key words and phrases. univalent function, circumferentially mean p-valent function, subordination and quasi-subordination, spirallike functions, close-to-convex functions, functions with positive real part, linear-invariant family.

1. Introduction

Let S denote the class of analytic functions $g(z) = z + c_2z^2 + \dots + c_nz^n + \dots$ that are univalent in the unit disk D $\{z: |z| < 1\}$ and normalized so that $g(0) = 0$, $g'(0) = 1$. By S_β, S^* and C we denote the sub-classes of S whose elements g(z) are respectively β - spirallike, starlike, close-to-convex in D. In particular $g \varepsilon S_\beta$ if, and only if, g is analytic in D, $g(0) = 0$, $g'(0) = 1$, and for some real β

$$\mathcal{Re}\left[e^{i\beta}\frac{zg'(z)}{g(z)}\right] > 0, \ z \varepsilon D.$$

Then $S^* = S_o$. Furthermore, $g \varepsilon C$ if, and only if, there exists a function $s(z) \varepsilon S^*$ and a complex constant a such that

$$\mathcal{Re}\left[a\frac{zg'(z)}{s(z)}\right] > 0, \ z \varepsilon D, \ g(0) = 0, \ g'(0) = 1.$$

For each of these subclasses of S it is well known that the Bieberbach coefficient inequalities hold:

$$|c_n| \leq n, \ n = 2, 3, \dots \quad . \tag{1.1}$$

In this paper we introduce a new class of univalent functions for which the inequalities that correspond to (1.1) are a matter of conjecture.

We shall be concerned with analytic functions f(z) that can be represented as a complex power, $f(z) = [F(z)]^x$, of an analytic function F(z) that is univalent and does not vanish in D. We normalize F(z) so that $F(0) = 1$ and choose for f(z) the branch for which $\log F(0) = 0$. We seek sufficient conditions on F(z) and the complex constant x so that $f(z) = [F(z)]^x = \exp[x \log F(z)]$ is also univalent or p-valent in D.

If x is real and $|x| > 1$ there exist univalent functions F(z) for which $f(z) = [F(z)]^x$ is not univalent in D. For example, $F(z) = (1 + z)^2$ is univalent and does not vanish in D, but for $x > 1$ $f(z) = (1 + z)^{2x}$ is not univalent in D, nor is $(1 + z)^{-2x}$. When x is not real more can be claimed as the following theorem illustrates.

<u>Theorem</u> A. <u>Let</u> x <u>be any non-real complex constant</u>. <u>Then there exists an analytic function</u> F(z), <u>dependent on</u> arg x <u>with</u> F(0) = 1, <u>univalent and not zero in</u> D, <u>such that</u> $f(z) = [F(z)]^x$, $f(0) = 1$, <u>is not p-valent in</u> D <u>for any positive integer</u> p.

Proof: We make use of the fact that the function $W(z) = (1 + z)^{bi}$, $W(0) = 1$, b real, has infinite valency in D. Indeed, W(z) takes on the value 1 at the points $z = \exp\left(-\frac{2n\pi}{|b|}\right) - 1$, $n = 0, 1, 2, \ldots$, which are in D. If $b = 0$, $W(z) = 1$ constantly.

Let x be a fixed complex number with $\mathcal{I}\, x > 0$. We write $x = i\,|x|e^{\gamma i}$, $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$. The function $F(z) = (1 + z)^{\mu}$ is univalent in D when $\mu = 1 + e^{-2\gamma i}$ since by Royster's test [19] $|\mu - 1| = 1$ and $\mu \neq 0$. The function $f(z) = [F(z)]^x = (1 + z)^{\mu x}$ has $\mathcal{R}e(\mu x) = 0$. Thus f(z) has infinite valency in D. If $\mathcal{I}\, x < 0$ we have only to replace x by -x and μ by $-\mu$ in this example. F(z) obviously satisfies all the conditions of the theorem. It depends on μ which depends only on the argument of x. This completes the proof.

At the other extreme there are functions F(z), dependent on x with $F(0) = 1$, univalent and not zero in D, such that $f(z) = [F(z)]^x$ is also univalent in D for every x for which $\mathcal{R}e\, x \neq 0$. For example, if $F(z) = (1 + z)^{\frac{1}{n}}$, $n > 1$, $f(z) = (1 + z)^{\frac{x}{n}}$, then f(z) is univalent in D for $|x - n| \leq n$ and n may be taken arbitrarily large.