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Maximum Principles on Riemannian Manifolds and Applications

Dedicated to the memory of
Franca Burrone Rigoli

Abstract

The aim of the paper is to introduce the reader to various forms of the maximum principle, starting from its classical formulation up to generalizations of the Omori-Yau maximum principle at infinity recently obtained by the authors. Applications are given to a number of geometrical problems in the setting of complete Riemannian manifolds, under assumptions either on the curvature or on the volume growth of geodesic balls.

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0. Introduction

The classical maximum principle in an invaluable tool in the study of the qualitative behavior of solutions of PDE's on domains of $\mathbb{R}^m$ or on $\mathbb{R}^n$ itself. Due to its local nature, it can be successfully applied on general Riemannian manifolds, to investigate equations of geometrical interests such as, for instance, the Yamabe equation, or the assigned mean curvature equation.

However, precisely because of its local nature, the maximum principle in not sensitive to the specific geometric properties of the manifold M. More than thirty years ago, H. Omori, studying immersions of minimal submanifolds into cones of $\mathbb{R}^m$, introduced a global version of the maximum principle, which has its roots in the following simple observation: if $u : \mathbb{R} \to \mathbb{R}$ is bounded above and we denote $u^* = \sup u$, then there exists a sequence $\{x_k\} \subset \mathbb{R}$ such that $u(x_k) \to u^*$, $u'(x_k) \to 0$ and $u''(x_k) \leq 1/k$, for every k. We refer to Section 1 below for a detailed discussion. Omori established a version of this principle on a Riemannian manifold with sectional curvature is bounded from below, and, perhaps more importantly, he also provided examples of manifolds where his global form of the maximum principle fails.

This new idea was taken up by S.T. Yau who, in a series of papers (some in collaboration with S.Y. Cheng), refined the principle for the Laplace–Beltrami operator, and applied it to find elegant solutions to a series of geometric problems, most notably, the Schwarz lemma for holomorphic maps between Kähler manifolds, that had eluded the efforts of many geometers for quite a few years.

This new maximum principle, which we will call henceforth the "Omori-Yau maximum principle", opened the way to a number of problems, which we may broadly collect into three categories.

1. Find a sharp form of the the maximum principle in relation with the geometry of the manifold;
2. Extend the maximum principle to differential operators other than the Laplacian;
3. Introduce some relaxed form of the maximum principle.

It turns out that the really challenging tasks are those described in the second and third point. Indeed, as far as the second point is concerned, we note that Yau's proof makes an essential use of the structural properties of the Laplacian, and it cannot carried out for instance, in the case of the mean curvature operator. Thus a genuinely new approach is needed.

As for the third point, the need for a relaxed form of the principle is suggested by a number of different geometrical problems in which the property that $|\nabla u(x_k)| \to 0$ plays no essential role.

The possibly unexpected fact is that this last requirement gives the link with curvature. It follows that, if we are not interested in a conclusion involving $|\nabla u|$, we should be able to relax the geometrical assumptions. This turns out to be the case, and, quite surprisingly, in the case of the Laplace–Beltrami operator, the validity of this new form of the maximum principle - we shall call it "the weak maximum principle" - is equivalent to the stochastic completeness of the manifold. It should be pointed out that L. Karp has obtained in [**Ka**] some results in this direction.

The aim of this paper is to present some of the evolution of the maximum principle, justifying its "raison d'etre" with applications to a few geometrical problems. At the same time we improve on known results, and we explore new interesting phenomena.

The paper is organized into six chapters as follows:

1. Preliminaries and Some Geometrical Motivations.
2. Further Typical Applications of Yau's Technique.
3. Stochastic Completeness and the Weak Maximum Principle.
4. The Weak Maximum Principle for the φ-Laplacian.
5. φ-parabolicity and Some Further Results.
7. Curvature and The Maximum Principle for the φ-Laplacian.

Each chapter begins with a few introductory observations to guide the reader to the core of the matter keeping the exposition basically self contained. Only occasionally, we refer the reader to the original papers for further details. The paper ends with a rich, but by no means exhaustive, bibliography.

Contents

CHAPTER 1

Preliminaries and some geometric motivations

Let $u : [a, b] \to \mathbb{R}$ be a continuous function. Then u takes its maximum u^* at some point $x_0 \in [a, b]$. If $x_0 \in (a, b)$ and u has continuous second derivative near x_0 then

$$(1.1) \qquad \text{(i) } u'(x_0) = 0 \quad \text{and} \quad \text{(ii) } u''(x_0) \leq 0.$$

It follows easily that, if u satisfies a differential inequality of the type

$$(1.2) \qquad u'' + g(x)u' > 0 \quad \text{on } (a, b),$$

where g is any bounded function, then either $x_0 = a$ or $x_0 = b$. Note, however, that for the non-strict inequality

$$(1.3) \qquad u'' + g(x)u' \geq 0 \quad \text{on } (a, b)$$

the constant solutions $u \equiv c$ is admitted and for such a solution the maximum is attained at any point of $[a, b]$. The content of the usual maximum principle is the fact that this exception is the only possible. The argument to prove this is a tricky way to pass from inequality (1.3) to inequality (1.2) for a new function v properly related to u. Then one concludes with the aid of the previous discussion. Thus, the core of the maximum principle indeed relies on $u(x_0) = u^*$ and (1.1) i), ii). Substituting $[a, b] \subset \mathbb{R}$ with a compact Riemannian manifold $(M, <, >)$ without boundary, we have that, given any $u \in C^2(M)$, there exists $x_0 \in M$ such that

$$(1.4) \qquad \text{(i) } u^* = u(x_0); \quad \text{(ii) } |\nabla u(x_0)| = 0, \quad \text{(iii) } \Delta u(x_0) \leq 0$$

where we have chosen to generalize (1.1) ii) with $\Delta u(x_0) \leq 0$; clearly we could have equally well considered as plausible (stronger) generalization, the following

$$(1.5) \qquad \text{(i) } u^* = u(x_0); \quad \text{(ii) } |\nabla u(x_0)| = 0; \quad \text{(iii) } \mathrm{Hess}(u)(x_0) \leq 0,$$

where the third condition has to be interpreted in the sense of quadratic form that is

$$\mathrm{Hess}(u)(x_0)(X, X) \leq 0, \quad \forall X \in T_{x_0}M.$$

Following S.T. Yau, from now on, we shall call "the usual maximum principle" (equivalently, "the finite maximum principle") the validity of either (1.4) or (1.5). Here are two examples of their use.

First example. Let $f : M \to \mathbb{R}^{m+1}$ be an immersed hypersurface and identify T_pM, for $p \in M$, with f_*T_pM viewed as an affine subspace of $\mathbb{R}^{m+1}$ passing through $f(p)$. Set

$$W = \mathbb{R}^{m+1} \setminus \bigcup_{p \in M} T_pM.$$

THEOREM 1.1. *Let $f : M \to \mathbb{R}^{m+1}$ be an oriented, minimally immersed hypersurface. If W is open and non-empty then f is totally geodesic, that is, $f(M)$ is a part of a hyperplane of $\mathbb{R}^{m+1}$.*

PROOF. We fix a point $o \in W$ as the origin of $\mathbb{R}^{m+1}$. For each $p \in M$, let $\nu(p)$ be the unit normal to $f(M)$ at $f(p)$ such that $\langle f(p), \nu(p)\rangle > 0$. This gives an orientation to M, indeed, the component of the position vector f perpendicular to M defines a never zero, normal, vector field on M, such that the support function $u = \langle f(p), \nu(p)\rangle$ is positive on M. We shall compute Δu. To this end, we choose an oriented Darboux frame along f, $(e_1, ..., e_m, \nu)$, so that the e_i 's, $i = 1, ..., m$, are tangent to M, give its orientation and $(e_1, ..., e_m, \nu)$ gives the canonical orientation of $\mathbb{R}^{m+1}$. Thus, in standard moving frame notations,

$$du = \langle df, \nu\rangle + \langle f, d\nu\rangle = (\langle e_i, \nu\rangle - h_{ki}\langle f, e_k\rangle)\,\theta^i = u_i\theta^i$$

where $\theta^i(e_j) = \delta^i_j$ and h_{ki} are the coefficients of the second fundamental form of f. Hence, setting θ^t_i for the connection forms of $(M, \langle,\rangle)$, we have

$$\begin{aligned} u_{ij}\theta^j &= du_i - u_t\theta^t_i = \langle de_i, \nu\rangle + \langle e_i, d\nu\rangle - h_{ki}\langle df, e_k\rangle - h_{ki}\langle f, de_k\rangle + \\ &- \langle f, e_k\rangle\, dh_{ki} - u_t\theta^t_i \\ &= -\left(h_{ji} + uh_{kj}h_{ki} + \langle f, e_k\rangle\, h_{kij}\right)\theta^j. \end{aligned}$$

It follows that, indicating with $|II|^2$ the square of the length of the second fundamental form of f,

$$\Delta u = -u\,|II|^2 - h_{ii} - \langle f, e_k\rangle\, h_{kii}.$$

However, minimality of f yields $h_{ii} = h_{kii} = 0$ so that we finally obtain

$$\Delta u + |II|^2\, u = 0, \quad \text{on } M \tag{1.6}$$

showing that $u > 0$ is super-harmonic on M. Let $u_* = \inf_M u$. If u_* is attained at some point $x_0 \in M$ then from the usual maximum principle u is constant, $u = u_* = u(x_0) > 0$. From (1.6) we then have $II \equiv 0$ and f is totally geodesic. Therefore the proof will be complete if we show the existence of x_0 such that $u_* = u(x_0)$. Towards this aim, we consider a sequence $\{x_k\} \subset M$ such that $u(x_k) \to u_*$ as $k \to +\infty$. To each x_k we associate y_k given by the intersection of $T_{x_k}M$ with the line perpendicular to such space and passing through o. Since $||y_k - o||_{\mathbb{R}^{m+1}} = u(x_k)$ is bounded, there exists a subsequence, which again we call $\{y_k\}$, such that $y_k \to y_0$ for some $y_0 \in \mathbb{R}^{m+1}$. Since $\bigcup_{p\in M} T_pM$ is closed and $\{y_k\} \subset \bigcup_{p\in M} T_pM$ we deduce $y_0 \in T_{x_0}M$ for some $x_0 \in M$. Thus,

$$u_* = \lim_{k\to+\infty} u(x_k) = \lim_{k\to+\infty} ||y_k - o||_{\mathbb{R}^{m+1}} = ||y_0 - o||_{\mathbb{R}^{m+1}} = u(x_0)$$

as needed. □

REMARK 1.2. If $m = \dim M = 2$ the theorem holds in the only assumptions that $(M, \langle,\rangle)$ is complete and $W \neq \emptyset$. Indeed, using Gauss equations and minimality of f we have $|II|^2 = -2K$, where K is the Gaussian curvature of M, and therefore u is a positive solution of

$$\Delta u - 2Ku = 0, \quad \text{on } M.$$

Using a highly non-trivial result of Fisher-Colbrie and Schoen, [**FCS**, Corollary 4], we can conclude that $II \equiv 0$.

REMARK 1.3. Theorem 1.1 has been proved in [**AF**]. The case $m = 2$ is older and due to [**HK**].

Second example. A theorem of Tompkins, later extended by Chern and Kuiper, asserts that a compact, flat, n-dimensional Riemannian manifold cannot be isometrically immersed in the standard $(2n-1)$-dimensional Euclidean space $(\mathbb{R}^{2n-1}, can)$ (See Corollary 1.24 and Theorem 1.15 for the Tompkins-Chern-Kuiper theorem and its generalization to the non-compact setting). Here, we outline the proof of the following conformal version of Tompkins result due to Moore, [**Mo3**].

THEOREM 1.4. *A flat, n-dimensional, compact, Riemannian manifold cannot be conformally immersed in $\mathbb{R}^{2n-2}$.*

PROOF. By contradiction, suppose we are given a conformal immersion $f: M \to \mathbb{R}^{m-1}$, where $(M, \langle , \rangle_M)$ is a flat, compact Riemannian manifold of dimension $\dim M = n$ and $m = 2n-1$. Since $\mathbb{R}^{m-1}$ is conformally diffeomorphic to the punctured standard sphere $\mathbb{S}^{m-1} \backslash \{point\} \subset \mathbb{R}^m$ endowed with its canonical metric $can_{\mathbb{S}^{m-1}}$, then we can consider f as a conformal immersion $f: M \to \mathbb{S}^{m-1}$.

The Euclidean space $\mathbb{R}^m$, hence $\mathbb{S}^{m-1}$, imbeds isometrically in the Lorentz space $\mathbb{L}^{m+1}$. This latter is nothing but the vector space $\mathbb{R}^{m+1}$ equipped with the inner product of signature $(1, m)$ given by

$$\langle v, w \rangle_{\mathbb{L}^{m+1}} = \sum_{j=1}^{m} v_j w_j - v_{m+1} w_{m+1}$$

for each $v = (v_1, ..., v_{m+1}), w = (w_1, ..., w_{m+1}) \in \mathbb{L}^{m+1}$. It is easily seen that the map

$$j : \mathbb{R}^m \to \mathbb{L}^{m+1} \quad \text{such that} \quad j(x) = (x, 1)$$

realises the isometric imbedding mentioned above. As a matter of facts, we have

$$\langle j(x), j(x) \rangle_{\mathbb{L}^{m+1}} = 0, \quad \forall x \in \mathbb{S}^{m-1} \subset \mathbb{R}^m$$

and hence j sends isometrically the sphere $\mathbb{S}^{m-1}$ into (a hypersurface of) the upper m-dimensional light cone $\mathbb{V}^m_+ \subset \mathbb{L}^{m+1}$:

$$\mathbb{V}^m_+ = \left\{ v \in \mathbb{L}^{m+1} : \langle v, v \rangle_{\mathbb{L}^{m+1}} = 0 \text{ and } v_{m+1} > 0 \right\}.$$

We recall that, if we define the lower light cone $\mathbb{V}^m_-$ in the obvious way, then a vector $v \in \mathbb{L}^{m+1}$ is said to be light-like if $v \in \mathbb{V}^m_\pm$; time-like if v lies "strictly inside" $\mathbb{V}^m_\pm$, that is, $\langle v, v \rangle_{\mathbb{L}^{m+1}} < 0$; space-like otherwise.
Moreover, if $v \in \mathbb{L}^{m+1}$ is either light-like or time-like then v is future pointing (resp. past pointing) if it lies "inside" $\mathbb{V}^m_+$ (resp. $\mathbb{V}^m_-$).

We now turn to the proof of Theorem 1.4. Starting from the conformal immersion $f: M \to \mathbb{S}^{m-1}$ we can construct an isometric immersion $g: M \to \mathbb{L}^{m+1}$ with $g(M) \subseteq \mathbb{V}^m_+$ as follows. Set $0 < \lambda \in C^\infty(M)$ for the conformality factor induced by f, that is, $f^* can_{\mathbb{S}^{m-1}} = \lambda^2 \langle , \rangle_M$ and define

$$g(x) = \frac{j \circ f(x)}{\lambda(x)}, \quad x \in M.$$

The fact that g is isometric enable us to consider its (Lorentzian) second fundamental form $II_p = \nabla dg : T_pM \times T_pM \to (T_pM)^\perp \subset \mathbb{L}^{m+1}$, for each $p \in M$. Since, by assumptions, $(M, \langle , \rangle_M)$ is flat (the Lorentzian version of) the Gauss equations yield, for each $p \in M$ and for each $X, Y, Z, W \in T_pM$,

$$\langle II_p(X,Y), II_p(Z,W) \rangle_{\mathbb{L}^{m+1}} - \langle II_p(X,W), II_p(Z,Y) \rangle_{\mathbb{L}^{m+1}} = 0.$$

According to a definition of Moore, [**Mo1**], this means that II_p is a flat bilinear form with respect to the Lorentz inner product $\langle , \rangle_{\mathbb{L}^{m+1}}$. We can therefore apply the machinery of flat bilinear forms (see [**Mo1**] and [**Mo2**]). Since $m = 2n - 2$, i.e. $\dim T_pM \geq \dim (T_pM)^{\perp}$, using also that $g(M) \subset \mathbb{V}_+^m$, we can find an orthonormal basis $\{e_1(p), ..., e_n(p)\}$ of T_pM and a future pointing light-like vector $e(p) \in (T_pM)^{\perp}$ such that

$$[II_p(e_i(p), e_j(p))] = \begin{bmatrix} A(p) & 0 \\ 0 & B(p) \end{bmatrix}$$

where $A(p)$ and $B(p)$ are square vector valued matrices of order $(m - n - 1)$ and $(2n - (m - 1))$, respectively; furthermore B satisfies

$$\begin{bmatrix} * & . & . & 0 \\ . & * & & . \\ . & & * & . \\ 0 & . & . & e(p) \end{bmatrix}.$$

Observe that, having fixed any future pointing time-like vector $v \in \mathbb{L}^{m+1}$, it holds $\langle v, e(p) \rangle_{\mathbb{L}^{m+1}} < 0$. Therefore, for each $p \in M$, the real-valued, bilinear form $\langle II_p, v \rangle_{\mathbb{L}^{m+1}} : T_pM \times T_pM \to \mathbb{R}$ possesses the negative eigenvalue $\langle v, e(p) \rangle_{\mathbb{L}^{m+1}}$.

Now, consider the height function in the direction v

$$h = \langle g, v \rangle_{\mathbb{L}^{m+1}} : M \to \mathbb{R}.$$

Since M is compact, h attains its absolute minimum at some $\bar{p} \in M$. Standard computations show that

$$\operatorname{Hess}(h)(\bar{p}) = (\nabla d \langle g, v \rangle_{\mathbb{L}^{m+1}})(\bar{p}) = \langle \nabla dg, v \rangle_{\mathbb{L}^{m+1}}(\bar{p}) = \langle II_{\bar{p}}, v \rangle_{\mathbb{L}^{m+1}},$$

and, applying the usual maximum principle (for the Hessian), we deduce that $\langle II_{\bar{p}}, v \rangle_{\mathbb{L}^{m+1}}$ is positive semi-definite. But this contradicts the existence of the negative eigenvalue $\langle v, e(\bar{p}) \rangle_{\mathbb{L}^{m+1}}$, found above, and finishes the proof. □

REMARK 1.5. Dimensionwise the theorem is a best possible result since the n-dimensional Clifford torus is conformally imbedded in $\mathbb{R}^{2n-1}$.

REMARK 1.6. It has been noticed that the work of Elie Cartan on the classification of conformally deformable Euclidean hypersurfaces suggests that many results on isometric immersions might extend to the conformal realm once dimensions are adjusted and the other assumptions are replaced with the conformal counterparts. This feeling is confirmed e.g. in [**dCD**], [**DT**] and Theorem 1.4 is no exception. Holding this line, in view of the extension of Tompkins theorem to complete manifolds one could try to extend Moore's result accordingly.

Clearly, it is not always possible, given $u \in C^2(M)$ with $u^* < +\infty$ to find $x_0 \in M$ such that $u(x_0) = u^*$. Nevertheless, given for instance $u : \mathbb{R} \to \mathbb{R}$ with $u^* < +\infty$, it is a simple matter to realize the existence of a sequence $\{x_k\} \subset \mathbb{R}$ with the following properties:

$$\text{i) } u(x_k) > u^* - \frac{1}{k}; \quad \text{ii) } |u'(x_k)| < \frac{1}{k}; \quad \text{iii) } u''(x_k) < \frac{1}{k}$$

for each $k \in \mathbb{N}$. More generally, given $u : \mathbb{R}^n \to \mathbb{R}$, $u^* < +\infty$, there exists $\{x_k\} \subset \mathbb{R}^n$ such that

$$\text{i) } u(x_k) > u^* - \frac{1}{k}; \quad \text{ii) } |\nabla u(x_k)| < \frac{1}{k}; \quad \text{iii) } \Delta u(x_k) < \frac{1}{k} \tag{1.7}$$

for each $k \in \mathbb{N}$. Here is an elementary proof in the spirit above. Its main idea consists in considering a family of functions each of which attains a maximum at some point of M goes back to Ahlfors, [**A**], and will be repeatedly applied in the sequel.

We fix a sequence $\{\varepsilon_k\} \searrow 0^+$ and we set

$$u_i(x) = u(x) - |x|^2 \varepsilon_i.$$

Clearly, u_i takes its (absolute) maximum at some point $x_i \in \mathbb{R}^n$ where

$$\Delta u_i(x_i) \le 0 \quad \text{and} \quad |\nabla u_i(x_i)| = 0.$$

Since in $\mathbb{R}^n$, $\Delta |x|^2 = 2m$ we obtain

$$\text{(1.8)} \qquad \text{i) } \Delta u(x_i) \le 2m\varepsilon_i \quad \text{and} \quad \text{ii) } \nabla u(x_i) = 2\varepsilon_i x_i.$$

On the other hand,

$$u(x_i) - \varepsilon_i |x_i|^2 = u_i(x_i) \ge u_i(0) = u(0)$$

and therefore

$$\varepsilon_i |x_i|^2 \le u(x_i) - u(0) \le u^* - u(0) \le C$$

for some constant $C > 0$. It follows that

$$|x_i| \le \frac{C}{\sqrt{\varepsilon_i}}.$$

From (1.8) ii) we then deduce

$$|\nabla u(x_i)| \le C\sqrt{\varepsilon_i}.$$

To conclude we fix $\eta > 0$ arbitrarily. Then, there exists $y \in \mathbb{R}^n$ such that

$$u(y) > u^* - \eta.$$

We have

$$\begin{aligned} u_i(x_i) = u(x_i) - \varepsilon_i |x_i|^2 &\ge u_i(y) = u(y) - \varepsilon_i |y|^2 \\ &> u^* - \eta - \varepsilon_i |y|^2, \end{aligned}$$

that is,

$$\text{(1.9)} \qquad u(x_i) \ge u^* - \eta - \varepsilon_i |y|^2.$$

Next, we fix $k \in \mathbb{N}$ and $\eta = 1/(2k)$ and we choose $i = \bar{\imath}$ sufficiently large that

$$\text{(1.10)} \qquad \varepsilon_{\bar{\imath}}\ |y|^2 < \frac{1}{k}, \quad C\sqrt{\varepsilon_{\bar{\imath}}} < \frac{1}{k}, \quad 2m\varepsilon_{\bar{\imath}} < \frac{1}{k}.$$

Correspondingly to this choice we set

$$x_k = x_{\bar{\imath}}.$$

It is immediate to see that the sequence $\{x_k\}$ constructed in this way satisfies (1.7).

In the previous proof there are two important facts that need to be stressed. The first is the equality

$$\Delta |x|^2 = 2m$$

which is tightly related to the geometry of $\mathbb{R}^n$, and the second is the linearity of the Laplace-Beltrami operator for which we have been able to perform the following computation

$$\Delta u_i = \Delta u - \varepsilon_i \Delta |x|^2.$$

Thus, for instance, the above technique certainly cannot be applied to the mean curvature operator. Furthermore, we observe that, given u as above, we can always find a sequence $\{x_k\}$ in $\mathbb{R}^n$ such that (1.7) i), ii) hold true. This is a general fact that can be easily proved.

PROPOSITION 1.7. Let $(M, \langle,\rangle)$ be a Riemannian manifold and let $u \in C^2(M)$ be such that $u^* < +\infty$. Given $\varepsilon > 0$, let $y \in M$ satisfy $u(y) > u^* - \varepsilon^2$ and suppose that the closed ball $\overline{B_\varepsilon(y)}$ is compact. Then, there exists $x \in \overline{B_\varepsilon(y)}$ with the properties

$$\text{(1.11)} \qquad \text{i) } u(x) \geq u(y) \quad \text{and} \quad \text{ii) } |\nabla u(x)| \leq \varepsilon.$$

PROOF. Let γ be the maximal integral curve of ∇u such that $\gamma(0) = y$, defined for $a < t < b$, $a < 0 < b$, and let $\beta = \sup\{t < b : \gamma([0,t)) \subset B_\epsilon(y)\}$. By the general theory of continuation of solutions of ODEs (see, e.g., [**Bo**, Lemma 5.1 page 138]), γ can be continued until it lies in the compact set $\overline{B_\varepsilon(y)}$, and we deduce that either $\beta < +\infty$ and then $\gamma(\beta) \in \partial B_\epsilon(y)$, or $\beta = b = +\infty$ and γ lies entirely in $B_\epsilon(y)$. Let $\tau = \min\{1, \beta\}$, so that $\gamma([0,\tau]) \in \bar{B}_\epsilon(y)$. We claim that there exists $\bar{t} \leq \tau$ such that $|\nabla u(\gamma(\bar{t}))| \leq \epsilon$. Since γ is an integral curve of ∇u, clearly $u(\gamma(\bar{t})) \geq u(\gamma(0)) = u(y)$.

To prove the claim, assume by contradiction that $|\nabla u| > \epsilon$ on $\gamma([0,\tau])$. Since $\dot{\gamma}(t) = \nabla u(\gamma(t))$, and $|\nabla u| > \epsilon$ in $\bar{B}_\epsilon(y)$, denoting by $l(\gamma|_{[0,\tau]})$ the length of the arc $\gamma|_{[0,\tau]}$, we have

$$l(\gamma|_{[0,\tau]}) = \int_0^\tau |\nabla u(\gamma(t))|\, dt \geq \max\{\epsilon\tau, d(\gamma(0), \gamma(\tau))\} \geq \epsilon.$$

Also,

$$\frac{d}{dt} u(\gamma(t)) = \langle \nabla u(\gamma(t)), \dot{\gamma}(t) \rangle = |\nabla u(\gamma(t))|\, |\dot{\gamma}(t)|\,,$$

whence

$$\text{(1.12)} \qquad u(\gamma(\tau)) - u(y) = \int_0^\tau |\nabla u(\gamma(t))|\, |\dot{\gamma}(t)|\, dt > \epsilon l(\gamma|_{[0,\tau]}) \geq \epsilon^2,$$

and therefore $u(y) < u(\gamma(\tau)) - \epsilon^2 \leq u^* - \epsilon^2$, contradicting the assumption that $u(y) > u^* - \epsilon^2$. □

REMARK 1.8. Some request on $B_\varepsilon(y)$ have to be considered as suggested by the following example. Let $M = \mathbb{R}^2 \setminus \{0\}$ with its canonical metric. We consider $u(x) = u(|x|) = e^{-|x|}$. Then $u \in C^\infty(M)$, $u^* = 1 = \lim_{|x|\to 0} u(x)$. On the other hand, $|\nabla u(x)| = e^{-|x|} \to 1$ as $|x| \to 0$.

Of course it is possible to reformulate (1.7) on a Riemannian manifold $(M, \langle,\rangle)$ and, as we have just seen, if the manifold is, for instance, complete, we can always find a sequence $\{x_n\} \subset M$ satisfying (1.7) i), ii). The next example shows that in general there might be no sequences satisfying all the three conditions in (1.7) at the same time, and points to the fact that some geometrical conditions need to be imposed in order to obtain validity of the whole (1.7).

Let $(M, \langle,\rangle)$ be $\mathbb{R}^2$ with the metric, in polar coordinates,

$$\text{(1.13)} \qquad \langle,\rangle = dr^2 + g(r)^2 d\theta^2$$

with $d\theta^2$ the standard metric of S^1, $g \in C^\infty([0,+\infty))$, $g(r) > 0$ for $r > 0$ and

$$g(r) = \begin{cases} r & \text{on } 0 \leq r < 1 \\ r\,(\log r)^{1+\mu}\, e^{r^2(\log r)^{1+\mu}} & \text{on } r > 3 \end{cases}$$

for some positive constant μ. We note that the behavior of g near 0 guarantees that (1.13) can be smoothly defined on all of $\mathbb{R}^2$. Obviously $\langle,\rangle$ is complete. We define

$$u(x) = u(r(x)) + \int_0^{r(x)} g(s)^{-1} \int_0^s g(t)dt\, ds.$$

Then, $u \in C^2(M)$, $\Delta u \equiv 1$ on M, and, since $\mu > 0$, $u^* < +\infty$. In this case property (1.7) iii) cannot hold. Note that in this example the Gaussian curvature K and the volume growth of the geodesic ball B_r have the following asymptotic behavior,

$$K(r) = -\frac{g''}{g}(r) \sim -c^2 r^2 (\log r)^{2(1+\mu)} \quad \text{as } r \to +\infty$$

for some constant $c > 0$, and

$$vol(B_r) \sim \frac{1}{2} e^{r^2(\log r)^{1+\mu}} \quad \text{as } r \to +\infty.$$

The following result is a generalization of Cheng and Yau, [**CY**], [**Y1**].

THEOREM 1.9. *Let $(M,\langle,\rangle)$ be a Riemannian manifold and assume that there exists a non-negative C^2 function γ satisfying the following requirements*

$$\gamma(x) \to +\infty \quad \textit{as } x \to \infty \tag{1.14}$$

$$\exists A > 0 \textit{ such that } |\nabla\gamma| \leq A\gamma^{1/2} \textit{ off a compact set} \tag{1.15}$$

$$\exists B > 0 \textit{ such that } \Delta\gamma \leq B\gamma^{1/2}G(\gamma^{1/2})^{1/2} \textit{ off a compact set} \tag{1.16}$$

where G is a smooth function on $[0,+\infty)$ satisfying

$$\begin{array}{ll} \textit{i) } G(0) > 0 & \textit{ii) } G'(t) \geq 0 \quad \textit{on } [0,+\infty) \\ \textit{iii) } G(t)^{-1/2} \notin L^1(+\infty) & \textit{iv) } \limsup_{t\to+\infty} \frac{tG(t^{1/2})}{G(t)} < +\infty. \end{array} \tag{1.17}$$

Then, given any function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$, there exists a sequence $\{x_n\}_n \subset M$ with the properties

$$\textit{i) } u(x_k) > u^* - \frac{1}{k}; \quad \textit{ii) } |\nabla u(x_k)| < \frac{1}{k}; \quad \textit{iii) } \Delta u(x_k) < \frac{1}{k} \tag{1.18}$$

for each $k \in \mathbb{N}$. If, instead of (1.16) we assume that

$$\exists B > 0 \textit{ such that } \mathrm{Hess}(\gamma) \leq B\gamma^{1/2}G(\gamma^{1/2})^{1/2}\langle,\rangle \textit{ off a compact set} \tag{1.19}$$

in the sense of quadratic forms, we can strengthen conclusion (1.18) iii) to

$$\mathrm{Hess}(u)(x_k) < \frac{1}{k}\langle,\rangle.$$

We recall that condition (1.14) means that for each $\eta > 0$ there is a compact set $K = K(\eta) \subset M$ such that $\gamma(x) > \eta$ whenever $x \notin K$.

DEFINITION 1.10. We say that the Omori–Yau maximum principle holds on $(M,\langle,\rangle)$ if the conclusion (1.18) of the Theorem is valid. In the case where the stronger statement concerning the Hessian is satisfied, we say that the Omori–Yau maximum principle for the Hessian holds on $(M,\langle,\rangle)$.

PROOF. We define the function

$$\varphi(t) = e^{\int_0^t G(s)^{-1/2} ds}$$

and note that $\varphi(t)$ is well defined, smooth, positive and it satisfies

$$\varphi(t) \to +\infty \quad \text{as } t \to +\infty. \tag{1.20}$$

We record, for future use, that

$$\varphi'(t) = G(t)^{-1/2}\varphi(t), \quad \varphi''(t) \le G(t)^{-1}\varphi(t)$$

and therefore,

$$\left(\frac{\varphi'(t)}{\varphi(t)}\right)^2 - \frac{\varphi''(t)}{\varphi(t)} \ge 0 \tag{1.21}$$

and, using assumption (1.17) iv),

$$\frac{\varphi'(t)}{\varphi(t)} \le c \, \left(tG(t^{1/2})\right)^{-1/2} \tag{1.22}$$

for some constant $c > 0$. Next, we fix a point $p \in M$ and, $\forall k \in \mathbb{N}$, we define

$$f_k(x) = \frac{u(x) - u(p) + 1}{\varphi(\gamma(x))^{1/k}}. \tag{1.23}$$

Then $f_k(p) = 1/\varphi(\gamma(p))^{1/k} > 0$. Moreover, since $u^* < +\infty$ and $\varphi(\gamma(x)) \to +\infty$ as $x \to \infty$, we have $\limsup_{x\to\infty} f_k(x) \le 0$. Thus, f_k attains a positive absolute maximum at $x_k \in M$. Iterating this procedure, we produce a sequence $\{x_k\}$. We begin by showing that

$$\limsup_{k\to+\infty} u(x_k) = u^*. \tag{1.24}$$

To prove the claim assume by contradiction that there exists $\hat{x} \in M$ such that

$$u(\hat{x}) > u(x_k) + \delta$$

for some $\delta > 0$ and for each $k \ge k_0$ sufficiently large. If $\gamma(x_k) \to +\infty$ as $k \to +\infty$, on a subsequence, for each k such that $\gamma(x_k) > \gamma(\hat{x})$ we have

$$f_k(\hat{x}) = \frac{u(\hat{x}) - u(p) + 1}{\varphi(\gamma(\hat{x}))^{1/k}} > \frac{u(x_k) - u(p) + 1 + \delta}{\varphi(\gamma(x_k))^{1/k}} > f_k(x_k)$$

contradicting the definition of x_k. If $\{x_k\}$ lies in a compact set, then up to passing to a subsequence, $\{x_k\} \to \bar{x}$ so that

$$u(\hat{x}) \ge u(\bar{x}) + \delta.$$

On the other hand, since $f_k(x_k) \ge f_k(\hat{x})$ for every k, we deduce that

$$u(\bar{x}) - u(p) + 1 = \lim_k f_k(x_k) \ge \lim_k f_k(\hat{x}) = u(\hat{x}) - u(p) + 1,$$

showing that

$$u(\bar{x}) \ge u(\hat{x}),$$

a contradiction. This proves (1.24) and, by passing to a subsequence if necessary, we may assume that

$$\lim_{k\to+\infty} u(x_k) = u^*. \tag{1.25}$$