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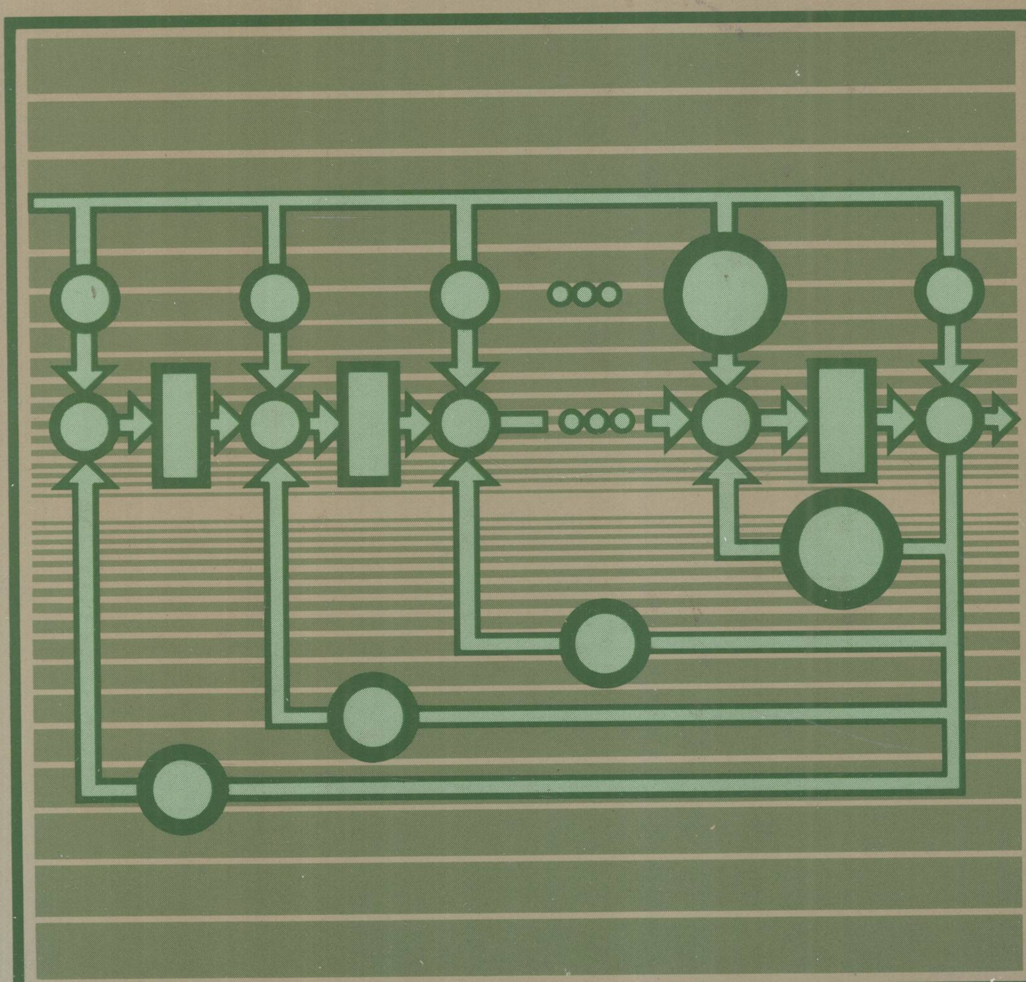
Time Domain and Transform Analysis



Linear Systems

TIME DOMAIN AND TRANSFORM ANALYSIS

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NOTATION IN ORDER OF ITS APPEARANCE

Chapter 1

LTIC — Linear, time-invariant causal (system).

$\sqcap(t)$ — The rectangle function:

$$\sqcap(t) = 1, \qquad -0.5 < t < 0.5; \qquad = 0, \qquad |t| < 0.5$$

$\delta(t)$ — The Dirac delta function:

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt \triangleq f(0), \qquad \text{for } f(t) \text{ continuous.}$$

$\delta(n)$ — The discrete pulse function or kronecker delta:

$$\delta(n) = 1, \qquad n = 0; \qquad = 0, \qquad \text{otherwise.}$$

$u(t)$, $u(n)$ — The continuous and discrete unit step functions:

$u(t) = 1, \quad t > 0; \quad = 0; \quad = 0, \quad t < 0,$
$u(n) = 1, \quad n \geq 0, \quad = 0, \quad n < 0$

$\delta^n(t)$ — $\int_{-\infty}^{\infty} f(t)\delta^n(t)\, dt = (-1)^n f^n(0)$

Chapter 2

LDECC — Linear differential or difference equation with constant coefficients:

$$a_n y^n(t) + a_{n-1}y^{n-1}(t) + \cdots + a_0 y(t) = f(t)$$

$$\text{or } a_n y(n) + a_{n-1}y(n-1) + \cdots + a_{n-p}y(n-p) = f(n)$$

$y_{h0}(t)$, $y_{h0}(n)$ — The homogeneous response found with $f(\,) = 0$.

$y_{f0}(t)$, $y_{f0}(n)$ — The forced response or the one unique solution of the LDECC that contains no part of y_{h0}.

SDE — The system differential or difference equation:

$$a_n y^n(t) + a_{n-1}y^{n-1}(t) + \cdots + a_0 y(t) = b_0 x(t) + \cdots + b_m x^m(t)$$

$$\text{or} \quad a_n y(n) \cdots + a_{n-p}y(n-p) = b_n x(n) + \cdots + b_{n-l}x(n-l)$$

$h(t)$, $h(n)$ — The impulse and unit pulse response $h(t)$ is the solution of the SDE with $x(t) = \delta(t)$ and $y(t) = 0$, $t < 0$; $h(n)$ is the solution of the SDE with $x(n) = \delta(n)$ and $y(n) = 0$, $n < 0$.

$H(s)$, $H(z)$ — The system function $y_{f0} \div x$ when $x(t) = e^{st}$ or $x(n) = z^n$:

$$H(s) = \frac{b_0 + bs + \cdots + b_m s^m}{a_2 + a_1 s + \cdots + a_n s^n}, \quad H(z) = \frac{b_n z^p + \cdots + b_0 z^{p-l}}{a_n z^p + \cdots + a_{n-p}}$$

$r_{xy}(t)$, $r_{xy}(n)$ or $x(\,)*y(\,)$ — The convolution of two deterministic functions x and y:

$$r_{xy}(t) = \int_{-\infty}^{\infty} x(p)y(t-p)\, dp, \quad r_{xy}(n) = \sum_{p=-\infty}^{\infty} x(p)y(n-p)$$

$y(t)$, $y(n)$ — The zero-state output of a LTIC system to $x(t)$ or $x(n)$, given as: $x(t)*h(t)$ or $x(n)*h(n)$

Chapter 3

$C_{xy}(\tau)$, $C_{xy}(n)$ — The cross-correlation of two deterministic, finite-energy waveforms x and y given as:

$$\int_{-\infty}^{\infty} x(p)y(p+\tau)\, dp \quad \text{or} \quad \sum_{-\infty}^{\infty} x(p)y(p+n)$$

$C_{xx}(\tau)$, $C_{xx}(n)$ — The case when $y = x$, the autocorrelation function

$R_{xx}(\tau)$, $R_{xx}(n)$ — The autocorrelation function for a finite-power periodic waveform or random process. As a time average:

$$R_{xx}(\tau) = \widetilde{x(t)x(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(p)x(p+\tau)\,dp$$

or $$R_{xx}(n) = \widetilde{x(p)x(p+n)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{p=-N}^{N} x(p)x(p+n)$$

As an ensemble average for a stationary random process:

$$R_{xx}(\tau) = \overline{x(t)x(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x_i(t)x_i(t+\tau)\,dt$$

$$R_{xx}(n) = \overline{x(p)x(p+n)} = \lim_{N\to\infty} \frac{1}{N} \sum_{l=1}^{N} x_i(p)x_i(p+n)$$

For ergodic processes, ensemble and time averages are identical.

$R_{xy}(\tau)$, $R_{xy}(n)$ — The cross-correlation function defined with $x(t+\tau)$ replaced by $y(t+\tau)$ and $x(p+n)$ replaced by $y(p+n)$ in the autocorrelation function definition.

$C_{hh}(\tau)$, $C_{hh}(n)$ — The correlation transfer function:

$$C_{hh}(\tau) = h(\tau) \oplus h(\tau) \qquad C_{hh}(n) = h(n) \oplus h(n)$$

$R_{yy}(\tau)$, $R_{yy}(n)$ — The autocorrelation function of the output of a LTIC system with random input:

$$R_{yy}(\tau) = R_{xx}(\tau) * C_{hh}(\tau) \qquad R_{yy}(n) = R_{xx}(n) * C_{hh}(n)$$

$R_{xy}(\tau)$, $R_{xy}(n)$ — The cross-correlation of a random input and output for a LTIC system, given as:

$$R_{xx}(\tau) * h(\tau) \quad \text{or} \quad R_{xx}(n) * h(n)$$

Chapters 4–7

$X(s)$ — The one- or two-sided Laplace transform, defined as:

$$\int_{0^-}^{\infty} x(t)e^{-st}\,dt, \quad \sigma > \sigma_1, \qquad \int_{-\infty}^{\infty} x(t)e^{-st}\,dt, \quad \sigma_1 < \sigma < \sigma_2$$

$X(z)$ — The one- or two-sided Z transform defined as:

$$\sum_{0}^{\infty} x(n)z^{-n}, \quad |z| > \rho_1, \qquad \sum_{-\infty}^{\infty} x(n)z^{-n}, \quad \rho_1 < |z| < \rho_2$$

$x(t)$ — The inverse Laplace transform, defined as:

$$x(t) = \frac{1}{2\pi j} \int_{\substack{\sigma-j\infty \\ C}}^{\sigma+j\infty} X(s)e^{st}\,ds$$

$$= \Sigma \text{ (residues of poles to the left of } \sigma), \qquad t > 0$$

$$= -\Sigma \text{ (residues of poles to the right of } \sigma), \qquad t < 0$$

(*Continues on inside back cover.*)

Linear Systems

Preface

The text *Linear Systems: Time Domain and Transform Analysis* contains material that is suitable for a first course in linear systems as well as for a more advanced course. The second course could be taught at either the first year graduate level or the advanced undergraduate level. Methods of representation and analysis are developed for both discrete time and continuous time systems. The parallel consideration of discrete and continuous ideas provides an efficient way to profit from the similarities between these two fundamental representational modalities.

The prerequisite skills for understanding this material are those that most engineering students have mastered by their junior year. In particular, a basic knowledge of differential equations and dc/ac circuit analysis is necessary. There are also other requirements that are more important but more difficult to gauge. These include the readers' sincere intention to learn and their dedication to the subject matter. Intention, rather than mere attention, implies an active involvement with the text. The level of involvement will vary for each student as he or she progresses through the book and will differ depending on the perspective brought to the material.

The order of presentation of the subject matter does not proceed at a uniform level of difficulty. This unevenness, we believe, actually mirrors the learning process itself. The first three chapters (those dealing with time-domain analysis) will probably be the hardest for the student to assimilate. Then, the degree of sophistication drops in Chapter 4 where we begin transform analysis. The level of difficulty is incremented with the Fourier transform (Chapter 8). The final chapter on state variables uses ideas from the entire text. In the actual knowledge acquisition process, time-domain analysis techniques are mastered first, providing an introduction to the rigor of linear system concepts. For most students this introduction occurs during their first circuits course. Solving differential or difference equations yields descriptions and predictions of important variables within the system under consideration. Although these solutions are often not easy to obtain, with transform techniques the process is reduced to mere algebra. Transforms not only provide interesting insights into problems but also make solutions easier to determine.

There are two major parts of the text plus a final culminating chapter on state variables. The first portion consists of time-domain analysis and comprises the first three chapters. Chapter 1 presents basic background material for studying linear systems: important operations on continuous and discrete waveforms and the theory of singularity functions. Chapter 2 deals with the response of linear time-invariant systems to known or deterministic inputs. As a prelude to finding such responses we will develop the ideas that:

1. A system is often governed by a differential or difference equation.
2. A continuous system can be characterized by the impulse response, $h(t)$, which is the response to a delta function $\delta(t)$, and a discrete system can be characterized by the pulse response, $h(n)$, which is the response to the unit pulse, $\delta(n)$.
3. The zero-state system output is given by the convolution of the impulse response with the input in the continuous case and by the convolution summation of the input with the pulse response in the discrete case.

Chapter 3 extends the material of Chapter 2 to the case of systems with random inputs or a signal plus noise input. This is a starred chapter denoted, Chapter 3*, which indicates it should be omitted in an introductory course. First, a detailed treatment of correlation integrals and summations for finite energy and finite power (periodic) functions is given. Then, using only the concept of an average value, correlation integrals and summations are defined and interpreted for simple random waveforms. The material is intuitively challenging and presents a novel introduction to the world of ergodic random processes. The chapter culminates with the derivation and application of the input–output relations for autocorrelation and cross-correlation functions for linear systems with random inputs.

The second part of the text consists of transform analysis and comprises Chapters 4 through 9. Prior to the middle 1970s, many students graduated with a knowledge of the one-sided Laplace transform and its application to solving for complete responses in *RLC* circuits. They also had a nodding acquaintance with the Fourier transform as an extension of the Fourier series. Now a graduating senior must have facility in using the Z transform as well as the Laplace and Fourier transforms. Concise and precise presentations are therefore required. Chapter 4 covers the one-sided Laplace transform with an emphasis on solving for linear systems with deterministic causal inputs. Chapter 5, a starred chapter, develops the two-sided Laplace transform and concentrates on applications involving systems with random or signal plus noise inputs. Chapter 6 treats the one-sided Z transform and stresses discrete systems with causal inputs. Its starred counterpart, Chapter 7, deals with the two-sided Z transform and in particular its use in analyzing discrete systems with random or signal plus noise inputs. The frequency interpretation is given for continuous and discrete signals by using the Fourier transform and discrete Fourier transform in Chapters 8 and 9. Chapter 8 considers Fourier series and develops the Fourier transform from the exponential Fourier series. At the end of the chapter a number of Fourier analysis applications are examined. In Chapter 9 the discrete Fourier transform is studied and the decimation in time and the decimation in frequency approaches to the fast Fourier transform (FFT) are discussed.

The third and final part of the text is a last long chapter, Chapter 10. This chapter deals with state variables and focuses specifically on applications in control theory. The material in Chapter 10 does not properly belong in either the time-domain analysis section or the transform analysis section of the text. However, it employs many of the ideas explored in the first nine chapters and can function as a culminating experience in the study of linear systems. The general state equation formulation is developed in various realizations for both the continuous and discrete cases. The solution of the state equations is considered. Controllability and observability are studied and state variable feedback is discussed along with the fundamentals of observer theory.

A goal of *Linear Systems: Time Domain and Transform Analysis* is to develop intuitive and practical understanding of the essentials in linear systems analysis. The stress is on fundamentals that are illustrated with many examples and problems. General theories are best learned through many particulars. The philosophy of "learning by doing" provides the framework for our presentations. Although we believe that there exists a rough proportionality between the amount of knowledge acquired and the number of problems worked, there are many different ways to use the text. We encourage instructors to experiment. The sequence that we use for the undergraduate and graduate courses is indicated schematically:

First Course	Second Course
• *Chapter 1* Signal Operations and Singularity Functions • *Chapter 2* Time-Domain Analysis of Linear Systems • *Chapter 4* The One-sided Laplace Transform • *Chapter 6* The One-sided Z Transform • *Chapter 8* The Fourier Transform • *Chapter 9* or *Chapter 10* (as time permits) The Fast Fourier Transform or State Variables	• Review of Laplace, Z, and Fourier Transforms • *Chapter 3** Linear Systems with Random Inputs • *Chapter 5** The Two-sided Laplace Transform • *Chapter 7** The Two-sided Z Transform • *Chapter 9* The Fast Fourier Transform • *Chapter 10* State Variables

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Michael O'Flynn
Eugene Moriarty

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*We suggest omitting starred chapters in an introductory course.

Chapter 1

Signal Operations and Singularity Functions

INTRODUCTION

Chapters 1 through 3 are concerned with the evaluation of the output of linear, time-invariant causal systems (LTIC) for different inputs. Both continuous and discrete systems are considered; they are subject to deterministic and random inputs.

A continuous system is one whose input $x(t)$ and output $y(t)$ are continuous time functions related by a rule as in Figure 1-1(a). A discrete system is one whose input $x(n)$ and output $y(n)$ are discrete time functions related by a rule as in Figure 1-1(b).

The case of systems with deterministic or known inputs is treated in Chapter 2, whereas random and signal plus random inputs are treated in Chapter 3. As a prerequisite to system analysis in both the time and transform domains (Chapters 2 through 9), it is essential to be able to:

1. represent both continuous and discrete signals
2. understand the important signal operations of time-scaling, reflecting, and time-shifting
3. physically interpret and intuitively and rapidly operate with singularity functions.

Tasks 1, 2, and 3 will be accomplished in Chapter 1.

1-1 CONTINUOUS AND DISCRETE WAVEFORMS

A continuous waveform $x(t)$ assigns a unique numerical value to $x(t)$ for all t, $-\infty < t < \infty$. A discrete waveform $x(n)$ assigns a unique numerical value to $x(n)$

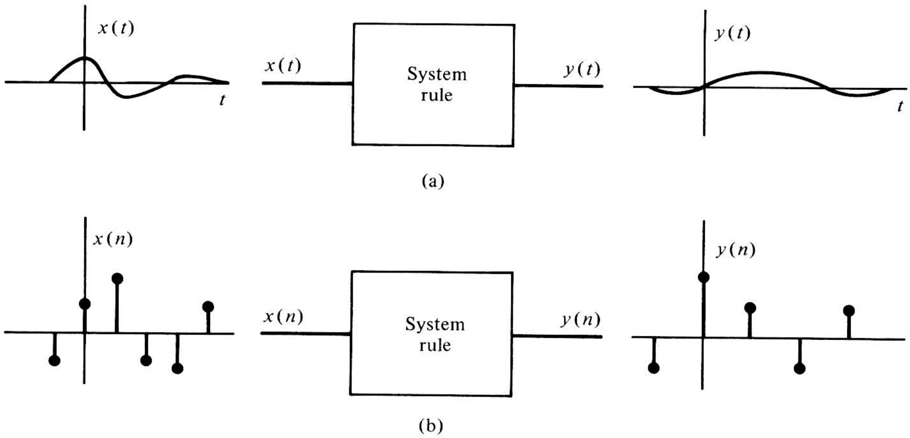

Figure 1-1 (a) A continuous system; (b) a discrete system.

for all integer n, $-\infty < n < \infty$. A number of waveforms that are prevalent throughout system theory will now be defined. A **waveform** is a function whose domain is from $-\infty$ to $+\infty$.

The Unit Step Function $u(t)$

$u(t)$ is defined as:

$$u(t) = 1, \qquad t > 0$$
$$= 0, \qquad t < 0$$

and its plot is shown in Figure 1-2(a).

The Rectangle Function $\sqcap(t)$

$\sqcap(t)$ is defined as:

$$\sqcap(t) = 1, \qquad -0.5 < t < 0.5$$
$$= 0, \qquad \text{otherwise, except at } t = -0.5 \text{ and } 0.5$$

The rectangle function is normalized with unit area and is even, as plotted in Figure 1.2(b).

The Triangle Function $\Lambda(t)$

$\Lambda(t)$ is defined as:

$$\Lambda(t) = 1 - |t|, \qquad -1 < t < 1$$
$$= 0, \qquad \text{otherwise}$$

The triangle function is normalized with unit area and is even, as shown in Figure 1-2(c).

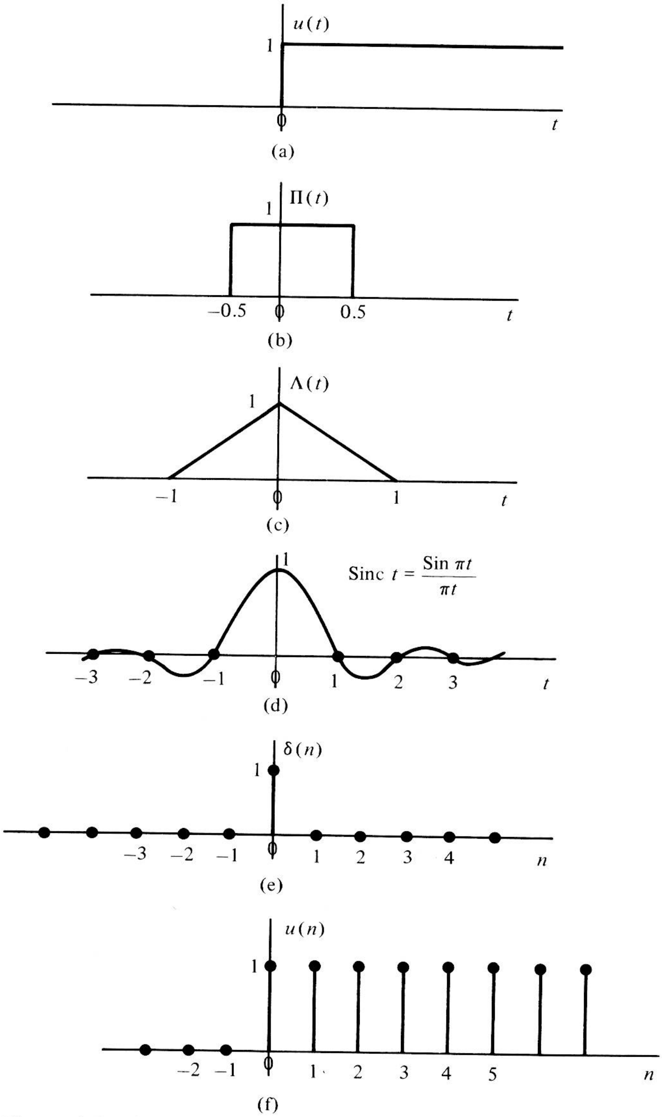

Figure 1-2 Some common functions of system theory.

The Sinc Function

Sinc (t) is defined as:

$$\text{Sinc }(t) = \frac{\text{Sin } \pi t}{\pi t}, \qquad -\infty < t < \infty$$

With investigation it can be shown (do so) that Sinc (t) has unit area, has a value of 1 at $t = 0$, and is even. The plot of Sinc (t) is illustrated in Figure 1.2(d).