Theory Of Block Designs

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A HALSTED PRESS BOOK

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Preface

The purpose of this book is to provide a systematic and self-contained account of construction, analysis and applications of block designs. The prerequisites for the book are an elementary knowledge of algebra of matrices and statistical distributions. The requirements in matrix algebra and statistical theory have however, been supplemented in Chapter 1, which deals with some important mathematical and statistical concepts and results.

The selection of material to cover the many developments in the theory and applications of block designs has not been easy. In making the present selection, I have been mainly guided by my own experience of teaching and, guiding and conducting research in design of experiments during the last fifteen years. Though most of the topics covered are fairly conventional, I hope the reader will find some novelty in the presentation of ideas. A fair portion of the book deals with construction and combinatorial aspects of block designs; however, I must emphasize that this is a book on Statistics and not on Combinatorial Mathematics.

The material of the book is presented in six chapters, including Chapter 1. A brief description of the contents of various chapters appears in the *Introduction.*

Certain results and sections are marked with an asterisk (*). The starred sections (or results) can easily be omitted in the first reading without disturbing the continuity of the text.

All chapters are appended with a set of problems which are meant to encourage the reader to go through the text again and practice, so as to become thoroughly familiar with the theory and methods described.

The book can be used in several ways. It can serve as a text at the master's level, if the starred sections and most of Chapter 6 is omitted. It will also be found useful by advanced level students and research workers as a reference book.

My sincere thanks go to several people for their help. I am deeply grateful to Dr. M.N. Das, who made many important suggestions to improve the first draft. Two of my friends, Drs. G.M. Saha and Bikas K. Sinha, deserve special thanks for reading through parts of the manuscript very critically and making numerous suggestions for improvement. I am also grateful to Prof. Prem Narain, Director, IASRI, for his encouragement and to the Indian Council of Agricultural Research for granting me permission to get this book published.

New Delhi ALOKE DEY
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Introduction

THE modern concepts of experimental designs are primarily due to Sir R.A. Fisher, who formulated and developed the basic ideas of statistical designing in the period 1919–30. Among the principles of design of experiments enunciated by Fisher are those of randomization, replication and local control or *blocking*. The concept of blocking in statistically planned experiments has its origin in the agricultural field experiments, conducted at the Rothamsted Experimental Station during the tenure of Fisher as the Chief Statistician. Fisher concluded that a completely random allocation of treatments over the experimental units would eliminate the bias in assessing treatment differences due to systematic variations. He then observed that it was possible to increase the sensitivity of the experiment by first grouping the experimental units (plots in agricultural experiments) so that the plots within a group or *block* were more or less homogeneous and then applying the treatments randomly to plots within a block. This concept led to the birth of randomized complete block designs or, rather, block designs in general. *Block design* is a design that is capable of eliminating heterogeneity (in the experimental material) in one direction.

Blocking of experimental units to eliminate heterogeneity is not restricted to agricultural experimentation alone. In the agricultural field experiments, experimental units lying at right angles to the fertility gradient generally form the blocks. Blocking of experimental units on a variety of physical, chemical, genetic, socio-economic, psychological or temporal characters have been adopted by various research workers. Discussion on blocking in actual situations may be found in most texts on Design of experiments, e.g., in Cochran and Cox (1957), Cox (1958), Kempthorne (1952) and Box *et al.* (1978).

Randomized complete block designs require that the common block size be equal to the number of treatments to be studied. This restriction initially did not create much problem for agricultural experiments. In many areas of research, however, the choice of block size is limited, and it is not always possible to find blocks of relatively homogeneous units of right size. Such situations are quite common in experiments with animals, taste-testing experiments, bio-assays, and many industrial experiments. These situations led to the development of incomplete block designs.

A large number of important contributions have been made in the area of block designs. These designs have immense applications in almost all areas of scientific investigation. Block designs have also opened up many interesting and challenging problems in combinatorial mathematics. In view of the statistical and mathematical importance of block designs, most books on Design of experiments deal with this topic. However, a full-length treatment of block designs has not so far appeared in the form of a book. The present volume intends to serve this purpose.

The text of the book is organized into six chapters. Chapter 1 is devoted to some important mathematical and statistical results. These results have been presented in a separate chapter, because the treatment in subsequent chapters depends heavily on these results.

The study of block designs is initiated in Chapter 2 by first describing the intra-block analysis of a general block design. The concept of balance is introducted next, and several results on balanced designs are discussed. Recovery of inter-block information and optimality of block designs are also dealt in this Chapter.

Balanced Incomplete Block (BIB) designs are introduced in Chapter 3. Some general properties of BIB designs are then discussed. Generalization of BIB designs to t-designs is also studied. Finally, the analysis of BIB designs with recovery of inter-block information is discussed at length. A numerical example is also given.

The construction and existence of BIB designs is the subject matter of study in Chapter 4. Several general methods of construction of BIB designs are discussed at length. Results on non-existence of BIB designs are also presented.

Partially Balanced Incomplete Block (PBIB) designs are covered in Chapter 5. The literature on PBIB designs is vast, and an attempt is made to present most of the important results systematically. Among the PBIB designs, the two-associate designs have been studied most extensively in the literature. Consequently, a large part of this chapter is devoted to the two-associate PBIB designs. The association schemes with more than two classes are described next. Some results on strongly regular graphs, non-existence of PBIB designs and analysis of PBIB designs are also covered.

Chapter 6, the last one, is concerned with several advanced topics in block designs. These include results on cyclic designs, *C*-designs, block designs with nested nuisance factors, block designs with factorial structure and designs for biological assays.

A list of references and a select bibliography is appended to each chapter to aid the reader in getting further insight in the theory of block designs. A number of problems are also included at the end of each chapter. It is believed that a genuine attempt to solve these problems will help the reader to get a better grasp of the subject matter.

References

Box, G.E.P., Hunter, W.G. and Hunter, J.S. (1978). *Statistics for Experimenters.* Wiley, New York.

Cochran, W.G. and Cox, G.M. (1957). *Experimental Designs,* second edn. Wiley, New York.

Cox, D.R. (1958). *Planning of Experiments.* Wiley, New York.

Kempthorne, O. (1952). *The Design and Analysis of Experiments.* Wiley, New York.

CHAPTER 1

Some Mathematical and Statistical Results

1.1 Introduction

THE purpose of this chapter is to introduce the reader to certain mathematical and statistical results, as these will be frequently used in the subsequent chapters. No attempt has been made for completeness and the interested reader is referred to the references at the end of the chapter for proofs and details. A certain amount of knowledge of the algebra of vectors and matrices as also elementary theory of statistical distributions is assumed throughout the book.

1.2 Groups, Fields and Some Results on Number Theory

(i) *Groups*. Let S be a set of elements $a, b, c, \ldots$. A *binary operation* (*) defined on the set S is an operation which, when operated on a pair of elements of S, leads to a unique element belonging to the set S, i.e., (*) is a binary operation if for $a, b \in S$, $a*b = g \in S$. The usual operations of addition and multiplication over the real number system are examples of binary operation.

Suppose we have a set S and a binary operation, (*). Consider the mathematical system $\mathscr{G} = \langle S, * \rangle$. The system is said to form a *group* if the following axioms are satisfied:

G 1. There exists an element $e \in S$ such that $e*x = x = x*e$, for all $x \in S$. The element e is called the *identity* of the group, and is unique.

G 2. For every element $x \in S$ there exists a unique element $x^{-1} \in S$, such that $x^{-1}*x = e = x*x^{-1}$. The element x^{-1} is called the *inverse* of the element x.

G 3. The binary operation (*) satisfies the associative law, i.e., if a, b, c are elements of S, then $a*(b*c) = (a*b)*c$.

In addition, if the following axiom is also satisfied, the group is called an *Abelian* group.

G 4. The binary operation satisfies the commutative law, i.e., if a and b are elements of S, then $a*b = b*a$.

Example 1.1 Let Z be the set of all integers, negative, zero and positive, and let (*) be the usual operation of addition. Then the system $\langle Z, *\rangle$ is a group with zero as the identity.

Example 1.2 Let Q be the set of rational numbers and let the binary operation (*) be addition. Then $\langle Q, *\rangle$ is a group with zero as the identity.

Example 1.3 Let Q^* be the set of non-zero rational numbers and the binary operation (*) be the usual multiplication of rational numbers. The system $\langle Q^*, *\rangle$ is a group with 1 as the identity.

Example 1.4 Let m be a fixed positive integer and let

$$S = \{0, 1, 2, \ldots, m-1\}.$$

Define a binary operation (*) as

$$\begin{aligned} a*b &= a+b, && \text{if } a+b<m \\ &= r, && \text{if } a+b=m+r \quad \text{for } 0 \leqslant r < m. \end{aligned}$$

It can be verified that $\langle S, *\rangle$ is a group, with zero as the identity.

(ii) *Subgroups.* Let $\langle S, *\rangle$ be *a* group, and let H be a nonempty subset of S. Then we say that H is a *subgroup* of S if the operation (*), restricted to H is a binary operation in H and makes $\langle H, *\rangle$ a group.

Let $\langle S, *\rangle$ be a group. Then a subset H of S is a subgroup of S if and only if,

(a) $H \neq \Phi$ (the null set) and, (b) if a, b are in H, then $a*b^{-1}$ is also in H.

Example 1.5 Let $\langle Q, +\rangle$ be the additive group of rationals and let Z be the set of integers (negative, zero, positive). Clearly $Z \neq \Phi$ and $Z \subset Q$ (Z is a subset of Q). If $a, b \in Z$, and the binary operation is $+$, the inverse of b is $-b$. Also $a+(-b) = a-b \in Z$. So $\langle Z, +\rangle$ is a subgroup of $\langle Q, +\rangle$.

The *order* of a group $\langle S, *\rangle$ is equal to the cardinality of the set S, i.e., order $= |S|$. In what follows, we will sometimes write ab for $a*b$.

(iii) *Cosets.* Let G be a group with respect to the operation (*), and H be a subgroup of G. Then a *right coset* of H in G is a subset of the form $H*g = \{x \mid x = h*g, h \in H\}$ for some g in G. We define a *left* coset of H in G to be a subset of the form $g*H = \{x \mid x = g*h, h \in H\}$.

A subgroup H of a group G is *normal* (or invariant) in G, if $g^{-1}*h*g \in H$ for all $g \in G$ and all $h \in H$. We write $H\Delta G$ to state that H is a normal subgroup of G.

Let G be a group and $N\Delta G$. Let us denote by G/N (the *factor group* of G by N) the set of right cosets of N in G. If we define a 'product' of two cosets Na and Nb as Nab, then under this operation G/N is a group.

(iv) *Cyclic groups.* A group H is said to be *cyclic* if we can find an element $x \in H$ such that $H = g\,p(x) = \{t \mid t = x^r, r \text{ an integer}\}$. Cyclic groups are necessarily Abelian. If G is a cyclic group of order m generated by the element x, then $G = \{x^0, x^1, \ldots, x^{m-1}\}$ and $x^m = 1$, where x^m is the least positive power of x, which is 1. The integer m is called the order of x. If x be of order $m < \infty$ and $x^r = 1$, then m divides r.

(v) *Fields.* Let F be a set of elements and let there be two binary operations, denoted by addition (+) and multiplication (·) signs. The system $\mathscr{F} = \langle F, +, \cdot \rangle$ is a field if it satisfies the following axioms:

F 1. $\langle F, + \rangle$ is an Abelian group whose identity will be denoted by 0 and the inverse of any element $x \in F$ will be denoted by $-x$.

F 2. $\langle F_0, \cdot \rangle$ is an Abelian group, where $F_0 = \{x \in F \mid x \neq 0\}$. The identity of this group will be denoted by 1 and the inverse of $x \in F_0$ (with respect to the operation 'multiplication') will be denoted by x^{-1}.

F 3. The operation multiplication is distributive over addition, i.e. $a.(b + c) = a.b + a.c$ for all $a, b, c \in F$.

If the set F has a finite number of elements, the field $\mathscr{F}$ is called a finite field or a *Galois field.* If $\mathscr{F} = \langle F, +, \cdot \rangle$ is a Galois field, then $|F| = p^n$ where p is a prime number and n ($\geqslant 1$) is a positive integer.

(vi) *Galois fields.* The quantity a is said to be *congruent* to b modulus m if $a - b$ is divisible by m. This fact is represented as

$a \equiv b \pmod{m}$. Congruences can be treated as equations for addition, subtraction and multiplication (but not for division). Thus, if $a \equiv b \pmod{m}$, then $ac \equiv bc \pmod{m}$ and $a \pm c \equiv b \pm c \pmod{m}$. Further, if $a \equiv b \pmod{m}$ and $d \equiv e \pmod{m}$, then $ad \equiv be \pmod{m}$, $a \pm d \equiv b \pm e \pmod{m}$. However, if $ac \equiv bc \pmod{m}$, we cannot say that $a \equiv b \pmod{m}$.

If $x \equiv a \pmod{m}$, then a is called the residue of x mod m. A class of residues mod m is the class of all integers congruent to a given residue (mod m) and every member of the class is called a representative of the class. There are in all m classes, denoted by (0), (1), (2), ..., $(m-1)$ and the representative 0, 1, 2, ..., $m-1$ of these classes are called a complete system of incongruent residues mod m.

In the complete system of incongruent residues mod m, we can define two binary operations, $(+_m)$ and $(\cdot_m)$ as

$$a\,(+_m)\,b = (a+b) \bmod m \tag{1.2.1}$$

$$a\,(\cdot_m)\,b = ab \bmod m, \tag{1.2.2}$$

where a, b are members of the complete system of residues mod m.

If $m = p$, a prime and $F_p = \{0, 1, 2, ..., p-1\}$, then the system $\mathscr{F}_p = \langle F_p, +_p, \cdot_p \rangle$ is a Galois field and is denoted by $GF(p)$. In fact, $\mathscr{F}_p$ is the simplest example of a Galois field.

If x is any element of $GF(p)$, we have from Fermat's theorem

$$x^{p-1} = 1 \tag{1.2.3}$$

where 1 is the representative of the class (1) and is the unit element of $GF(p)$. If r is the least positive integer such that $x^r = 1$, then r is called the *order* of the element x. When for certain $x = x'$, r has the maximum value $p-1$, x' is said to be a *primitive element* of $GF(p)$. There exists a primitive element in every $GF(p)$. If x is a primitive element, all the non-zero elements of $GF(p)$ can be expressed as

$$x^0 = 1, x, x^2, ..., x^{p-2} \tag{1.2.4}$$

which is called the *power cycle* of x. The power cycles for $p = 3$, 5, 7, 11, 13, 17, 19, 23, 31, 37 and 41 are given in Table 1.1.

A Galois field of p^n elements where n is a positive integer and p is prime, is obtained as follows : Let $P(x)$ be any polynomial in x of degree n with coefficients belonging to $GF(p)$ and $F(x)$, any polynomial in x with integral coefficients.

TABLE 1.1. Power cycles for some $GF(p)$

p	Primitive element	Power cycle																	
3	2	1,	2.																
5	2	1,	2,	4,	3.														
7	3	1,	3,	2,	6,	4,	5.												
11	2	1,	2,	4,	8,	5,	10,	9,	7,	3,	6.								
13	2	1,	2,	4,	8,	3,	6,	12,	11,	9,	5,	10,	7.						
17	3	1,	3,	9,	10,	13,	5,	15,	11,	16,	14,	8,	7,	4,	12,	2,	6.		
19	2	1,	2,	4,	8,	16,	13,	7,	14,	9,	18,	17,	15,	11,	3,	6,	12,	5,	10
23	5	1,	5,	2,	10,	4,	20,	8,	17,	16,	11,	9,	22,	18,	21,	13,			
		19,	3,	15,	6,	7,	12,	14.											
31	3	1,	3,	9,	27,	19,	26,	16,	17,	20,	29,	25,	13,	8,	24,	10,			
		30,	28,	22,	4,	12,	5,	15,	14,	11,	2,	6,	18,	23,	7,	21.			
37	2	1,	2,	4,	8,	16,	32,	27,	17,	34,	31,	25,	13,	26,	15,	30,			
		23,	9,	18,	36,	35,	33,	29,	21,	5,	10,	20,	3,	6,	12,	24,			
		11,	22,	7,	14,	28,	19.												
41	7	1,	7,	8,	15,	23,	38,	20,	17,	37,	13,	9,	22,	31,	12,				
		2,	14,	16,	30,	5,	35,	40,	34,	33,	26,	18,	3,	21,	24,				
		4,	28,	32,	19,	10,	29,	39,	27,	25,	11,	36,	6.						