DIFFERENTIAL & INTEGRAL CALCULUS

VOLUME II

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DIFFERENTIAL AND INTEGRAL CALCULUS

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VOLUME II

BLACKIE & SON LIMITED
LONDON AND GLASGOW

First published 1936
Reprinted 1937, 1940, 1941. 1942 (twice), 1943, 1944, 1945, 1947, 1948, 1949, 1950 1951, 1952, 1953, 1954 , 1956(twice), 1957(twice)

Printed in Great Britain by Blackie & Son, Ltd., Glasgow

PREFACE

The present volume contains the more advanced parts of the differential and integral calculus, dealing mainly with functions of several variables. As in Volume I, I have sought to make definitions and methods follow naturally from intuitive ideas and to emphasize their physical interpretations—aims which are not at all incompatible with rigour.

I would impress on readers new to the subject, even more than I did in the preface to Volume I, that they are not expected to read a book like this consecutively. Those who wish to get a rapid grip of the most essential matters should begin with Chapter II, and next pass on to Chapter IV; only then should they fill in the gaps by reading Chapter III and the appendices to the various chapters. It is by no means necessary that they should study Chapter I systematically in advance.

The English edition differs from the German in many details, and contains a good deal of additional matter. In particular, the chapter on differential equations has been greatly extended. Chapters on the calculus of variations and on functions of a complex variable have been added, as well as a supplement on real numbers.

I have again to express my very cordial thank to my German publisher, Julius Springer, for his generous attitude in consenting to the publication of the English edition. I have also to thank Blackie & Son, Ltd., and their staff, especially Miss W. M. Deans, for co-operating with me and my assistants and relieving me of a considerable amount of proof reading. Finally, I must express my gratitude to the friends and colleagues who have assisted me in preparing the manuscript for the press, reading the proofs, and collecting the examples; in the first place to Dr. Fritz John, now of the University of Kentucky, and to Miss Margaret Kennedy, Newnham College, Cambridge, and also to Dr. Schönberg, Swarthmore College, Swarthmore, Pa.

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New Rochelle, New York.
March, 1936.

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CHAPTER I

Preliminary Remarks on Analytical Geometry and Vector Analysis

In the interpretation and application of the mathematical facts which form the main subject of this second volume it is often convenient to use the simple fundamental concepts of analytical geometry and vector analysis. Hence, even though many readers will already have a certain knowledge of these subjects, it seems advisable to summarize their elements in a brief introductory chapter. This chapter, however, need not be studied before the rest of the book is read; the reader is advised to refer to the facts collected here only when he finds the need of them in studying the later parts of the book.

1. Rectangular Co-ordinates and Vectors

1. Co-ordinate Axes.

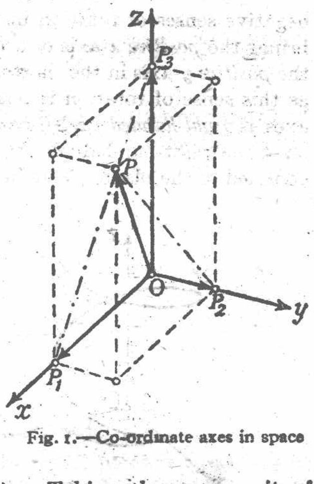

Fig. 1.—Co-ordinate axes in space

To fix a point in a plane or in space, as is well known, we generally make use of a rectangular co-ordinate system. In the plane we take two perpendicular lines, the x-axis and the y-axis; in space we take three mutually perpendicular lines, the x-axis, the y-axis, and the z-axis. Taking the same unit of length on each axis, we assign to each point of the plane an x-co-ordinate and a y-co-ordinate in the usual way, or to each point in space an x-co-ordinate, a y-co-ordinate, and a z-co-ordinate (fig. 1). Conversely, to every set of values (x, y) or (x, y, z) there corresponds just one point of the plane, or of space, as the case may be; a point is completely determined by its co-ordinates.

Using the theorem of Pythagoras we find that *the distance between two points* (x_1, y_1) *and* (x_2, y_2) *is given by*

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

(E912)

while the distance between the points with co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

In setting up a system of rectangular axes we must pay attention to the *orientation* of the co-ordinate system.

In Vol. I, Chap. V, § 2 (p. 268) we distinguished between positive and

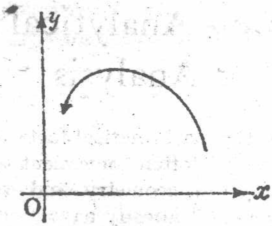

Fig. 2.—Right-handed system of axes

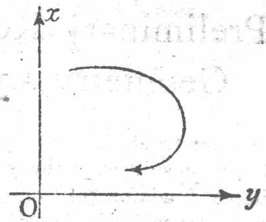

Fig. 3.—Left-handed system of axes

negative senses of rotation in the plane. The rotation through 90° which brings the *positive* x-axis of a plane co-ordinate system into the position of the *positive* y-axis in the shortest way defines a sense of rotation. According as this sense of rotation is positive or negative, we say that the system of axes is *right-handed* or *left-handed* (cf. figs. 2 and 3). It is impossible to change a right-handed system into a left-handed system by a rigid motion confined to the plane. A similar distinction occurs with co-ordinate systems

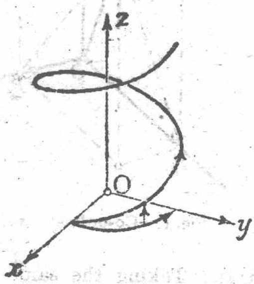

Fig. 4.—Right-handed screw

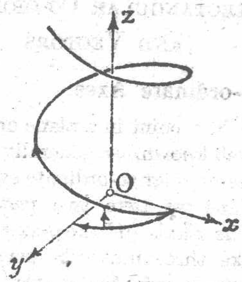

Fig. 5.—Left-handed screw

in space. For if one imagines oneself standing on the xy-plane with one's head in the direction of the positive z-axis, it is possible to distinguish two types of co-ordinate system by means of the apparent orientation of the co-ordinate system in the xy-plane. If this system is right-handed the system in space is also said to be right-handed, otherwise left-handed (cf. figs. 4 and 5). A right-handed system corresponds to an ordinary right-handed screw; for if we make the xy-plane rotate about the z-axis (in the sense prescribed by its orientation) and simultaneously give it a motion of translation along the positive z-axis, the combined motion is obviously that of a right-handed screw. Similarly, a left-handed system corresponds to a left-handed screw. No rigid motion in three dimensions can transform a left-handed system into a right-handed system.

In what follows we shall always use right-handed systems of axes.

We may also assign an orientation to a system of three arbitrary axes passing through one point, provided these axes do not all lie in one plane, just as we have done here for a system of rectangular axes.

2. Directions and Vectors. Formulæ for Transforming Axes.

An oriented line l in space or in a plane, that is, a line traversed in a definite sense, represents a *direction*; every oriented line that can be made to coincide with the line l in position and sense by displacement parallel to itself represents the same direction. It is customary to specify a direction relative to a co-ordinate system by drawing an oriented half-line in the given direction, starting from the origin of the co-ordinate system, and on this half-line taking the point with co-ordinates (α, β, γ) which is at unit distance from the origin. The numbers α, β, γ are called the *direction cosines* of the direction. They are the cosines of the three angles $\delta_1, \delta_2, \delta_3$ which the oriented line l makes with the positive x-axis, y-axis, and z-axis * (cf. fig. 6); by the distance formula, they satisfy the relation

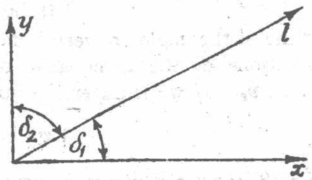

Fig. 6.—The angles which a straight line makes with the axes

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

If we restrict ourselves to the xy-plane, a direction can be specified by the angles δ_1, δ_2 which the oriented line l having this direction and passing through the origin forms with the positive x-axis and y-axis; or by the direction cosines $\alpha = \cos\delta_1$, $\beta = \cos\delta_2$, which satisfy the equation

$$\alpha^2 + \beta^2 = 1.$$

A line-segment of given length and given direction we shall call a *vector*; more specifically, a *bound vector* if the initial point is fixed in space, and a *free vector* if the position of the initial point is immaterial. In the following pages, and indeed throughout most of the book, we shall omit the adjectives free and bound, and if nothing is said to the contrary we shall always take the vectors to be free vectors. We denote vectors by heavy type, e.g. $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$, $\mathbf{A}$. Two free vectors are said to be equal if one of them can be made to coincide with the other by displacement parallel to itself. We sometimes call the length of a vector its absolute value and denote it by $|\mathbf{a}|$.

* The angle which one oriented line forms with another may always be taken as being between 0 and π, for in what follows only the cosines of such angles will be considered.

If from the initial and final points of a vector v we drop perpendiculars on an oriented line l, we obtain an oriented segment on l corresponding to the vector. If the orientation of this segment is the same as that of l, we call its length the *component of v in the direction of l*; if the orientations are opposite, we call the negative of the length of the segment the *component of v in the direction of l*. The component of v in the direction of l we denote by v_l. If δ is the angle between the direction of v and that of l (cf. fig. 7), we always have

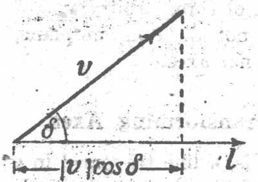

Fig. 7.—Projection of a vector

$$v_l = |v| \cos\delta.$$

A vector v of length 1 is called a *unit vector*. Its component in a direction l is equal to the cosine of the angle between l and v. The components of a vector v in the directions of the three axes of a co-ordinate system are denoted by v_1, v_2, v_3. If we transfer the initial point of v to the origin, we see that

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

If α, β, γ are the direction cosines of the direction of v, then

$$v_1 = |v|\alpha, \quad v_2 = |v|\beta, \quad v_3 = |v|\gamma.$$

A free vector is completely determined by its components v_1, v_2, v_3.

An equation

$$v = w$$

between two vectors is therefore equivalent to the three ordinary equations

$$v_1 = w_1,$$
$$v_2 = w_2,$$
$$v_3 = w_3.$$

There are two different reasons why the use of vectors is natural and

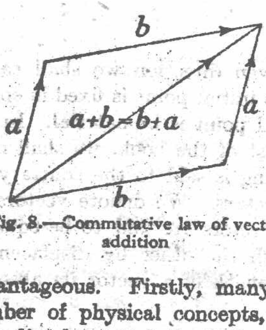

Fig. 8.—Commutative law of vector addition

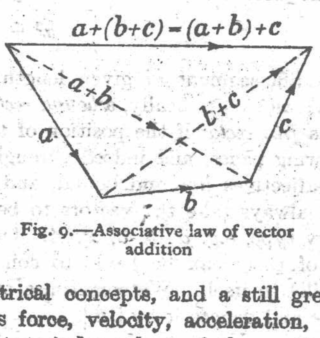

Fig. 9.—Associative law of vector addition

advantageous. Firstly, many geometrical concepts, and a still greater number of physical concepts, such as force, velocity, acceleration, &c., immediately reveal themselves as vectors independent of the particular co-ordinate system. Secondly, we can set up simple rules for calculating with vectors analogous to the rules for calculating with ordinary numbers; by means of these many arguments can be developed in a simple way, independently of the particular co-ordinate system chosen.

We begin by defining the *sum* of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. For this purpose we displace the vector $\boldsymbol{b}$ parallel to itself until its initial point coincides with the final point of $\boldsymbol{a}$. Then the initial point of $\boldsymbol{a}$ and the final point of $\boldsymbol{b}$ determine a new vector $\boldsymbol{c}$ (see fig. 8) whose initial point is the initial point of $\boldsymbol{a}$ and whose final point is the final point of $\boldsymbol{b}$. We call $\boldsymbol{c}$ the sum of $\boldsymbol{a}$ and $\boldsymbol{b}$ and write

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}.$$

For this additive process the *commutative law*

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

and the *associative law*

$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$$

obviously hold, as a glance at figs. 8 and 9 shows.

From the definition of vector addition we at once obtain the "projection theorem": *the component of the sum of two or more vectors in a direction l is equal to the sum of the components of the individual vectors in that direction*, that is,

$$(\boldsymbol{a} + \boldsymbol{b})_l = a_l + b_l.$$

In particular, the components of $\boldsymbol{a} + \boldsymbol{b}$ in the directions of the co-ordinate axes are $a_1 + b_1$, $a_2 + b_2$, $a_3 + b_3$.

To form the sum of two vectors we accordingly have the following simple rule. *The components of the sum are equal to the sums of the corresponding components of the summands.*

Every point P with co-ordinates (x, y, z) may be determined by the *position vector* from the origin to P, whose components in the directions of the axes are just the co-ordinates of the point P. We take three unit vectors in the directions of the three axes, $\boldsymbol{e}_1$ in the x-direction, $\boldsymbol{e}_2$ in the y-direction, $\boldsymbol{e}_3$ in the z-direction. If the vector $\boldsymbol{v}$ has the components v_1, v_2, v_3, then

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3.$$

We call $\boldsymbol{v}_1 = v_1\boldsymbol{e}_1$, $\boldsymbol{v}_2 = v_2\boldsymbol{e}_2$, $\boldsymbol{v}_3 = v_3\boldsymbol{e}_3$ the *vector components of* $\boldsymbol{v}$.

Using the projection theorem stated above, we easily obtain the *transformation formulæ* which determine (x', y', z'), the co-ordinates of a given point P with respect to the axes Ox', Oy', Oz', in terms of (x, y, z), its co-ordinates with respect to another set * of axes Ox, Oy, Oz which has the same origin as the first set and may be obtained from it by rotation. The three new axes form angles with the three old axes whose cosines may be

* It is to be noted that in accordance with the convention adopted on p. 3 both systems of axes are to be right-handed.

expressed by the following scheme, where for example γ_1 is the cosine of the angle between the x'-axis and the z-axis:

	x	y	z
x'	α_1	β_1	γ_1
y'	α_2	β_2	γ_2
z'	α_3	β_3	γ_3

From P we drop perpendiculars to the axes Ox, Oy, Oz, their feet being P_1, P_2, P_3 (cf. fig. 1, p. 1). The vector from O to P is then equal to the sum of the vectors from O to P_1, from O to P_2, and from O to P_3. The direction cosines of the x'-axis relative to the axes Ox, Oy, Oz are α_1, β_1, γ_1, those of the y'-axis α_2, β_2, γ_2, and those of the z'-axis α_3, β_3, γ_3. By the projection theorem we know that x', which is the component of the vector $\overrightarrow{OP}$ in the direction of the x'-axis, must be equal to the sum of the components of $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$, $\overrightarrow{OP_3}$ in the direction of the x'-axis, so that

$$x' = \alpha_1 x + \beta_1 y + \gamma_1 z,$$

for $\alpha_1 x$ is the component of x in the direction of the x'-axis, and so on. Carrying out similar arguments for y' and z', we obtain the *transformation formulæ*

$$\begin{aligned} x' &= \alpha_1 x + \beta_1 y + \gamma_1 z \\ y' &= \alpha_2 x + \beta_2 y + \gamma_2 z \\ z' &= \alpha_3 x + \beta_3 y + \gamma_3 z, \end{aligned}$$

and conversely

$$\begin{aligned} x &= \alpha_1 x' + \alpha_2 y' + \alpha_3 z' \\ y &= \beta_1 x' + \beta_2 y' + \beta_3 z' \\ z &= \gamma_1 x' + \gamma_2 y' + \gamma_3 z'. \end{aligned}$$

Since the components of a bound vector v in the directions of the axes are expressed by the formulæ

$$\begin{aligned} v_1 &= x_2 - x_1 \\ v_2 &= y_2 - y_1 \\ v_3 &= z_2 - z_1, \end{aligned}$$

in which (x_1, y_1, z_1) are the co-ordinates of the initial point and (x_2, y_2, z_2) the co-ordinates of the final point of v, it follows that the *same* transformation formulæ hold for the components of the *vector* as for the co-ordinates:

$$\begin{aligned} v_1' &= \alpha_1 v_1 + \beta_1 v_2 + \gamma_1 v_3 \\ v_2' &= \alpha_2 v_1 + \beta_2 v_2 + \gamma_2 v_3 \\ v_3' &= \alpha_3 v_1 + \beta_3 v_2 + \gamma_3 v_3. \end{aligned}$$

3. Scalar Multiplication of Vectors.

Following conventions like those for the addition of vectors, we now define the product of a vector v by a number c: if v has the components

v_1, v_2, v_3, then cv is the vector with components cv_1, cv_2, cv_3. This definition agrees with that of vector addition, for $v + v = 2v$, $v + v + v = 3v$, and so on. If $c > 0$, cv has the same direction as v, and is of length $c\,|v|$; if $c < 0$, the direction of cv is opposite to the direction of v, and its length is $(-c)\,|v|$. If $c = 0$, we see that cv is the *zero vector* with the components 0, 0, 0.

We can also define the *product* of two vectors u and v, where this "multiplication" of vectors satisfies rules of calculation which are in part similar to those of ordinary multiplication. There are two different kinds of vector multiplication. We begin with *scalar multiplication*, which is the simpler and the more important for our purposes.

*By the scalar product * uv of the vectors u and v we mean the product of their absolute values and the cosine of the angle δ between their directions*:

$$uv = |u|\,|v| \cos\delta.$$

The scalar product, therefore, is simply the component of one of the vectors in the direction of the other multiplied by the length of the second vector.

From the projection theorem the *distributive law* for multiplication,

$$(u + v)w = uw + vw,$$

follows at once, while the *commutative law*,

$$uv = vu,$$

is an immediate consequence of the definition.

On the other hand, there is an essential difference between the scalar product of two vectors and the ordinary product of two numbers, for *the product can vanish although neither factor vanishes*.

If the lengths of u and v are not zero, the product uv vanishes if, and only if, the two vectors u and v are perpendicular to one another.

In order to express the scalar product in terms of the components of the two vectors, we take both the vectors u and v with initial points at the origin. We denote their vector components by u_1, u_2, u_3 and v_1, v_2, v_3 respectively, so that $u = u_1 + u_2 + u_3$ and $v = v_1 + v_2 + v_3$. In the equation $uv = (u_1 + u_2 + u_3)(v_1 + v_2 + v_3)$ we can expand the product on the right in accordance with the rules of calculation which we have just established; if we notice that the products u_1v_2, u_1v_3, u_2v_1, u_2v_3, u_3v_1, and u_3v_2 vanish because the factors are perpendicular to one another, we obtain $uv = u_1v_1 + u_2v_2 + u_3v_3$. Now the factors on the right have the same direction, so that by definition $u_1v_1 = u_1v_1$, &c., where u_1, u_2, u_3 and v_1, v_2, v_3 are the components of u and v respectively. Hence

$$uv = u_1v_1 + u_2v_2 + u_3v_3.$$

This equation could have been taken as the definition of the scalar product, and is an important rule for calculating the scalar product of two vectors

* Often called the *inner product*.

given in terms of their components. In particular, if we take u and v as unit vectors with direction cosines $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ respectively, the scalar product is equal to the cosine of the angle between u and v, which is accordingly given by the formula

$$\cos\delta = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3.$$

The *physical* meaning of the scalar product is exemplified by the fact, proved in elementary physics, that a force f which moves a particle of unit mass through the directed distance v does work amounting to fv.

4. The Equations of the Straight Line and of the Plane.

Let a straight line in the xy-plane or a plane in xyz-space be given. In order to find their equations we erect a perpendicular to the line (or

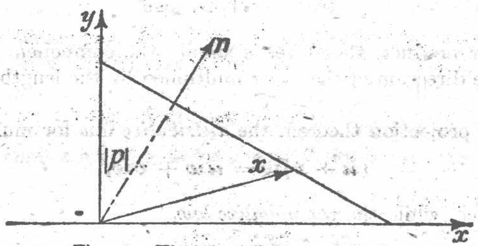

Fig. 10.—The equation of a straight line

the plane) and specify a definite "positive direction along the normal", perpendicular to the line (or plane); it does not matter which of the two possible directions is taken as positive (cf. fig. 10). The vector with unit length and the direction of the positive normal we denote by n. The points of the line (or plane) are characterized by the property that the position vector x from the origin to them has a constant projection p on the direction of the normal; in other words, the scalar product of this position vector and the normal vector n is constant. If α, β (or α, β, γ) are the direction cosines of the positive direction of the normal, that is, the components of n, then

$$\alpha x + \beta y - p = 0$$

$$(\text{or}\quad \alpha x + \beta y + \gamma z - p = 0)$$

is the required equation of the line (or plane). Here p has the following meaning: the absolute value $|p|$ of p is the distance of the line (or plane) from the origin. Moreover, p is positive if the line (or plane) does not pass through the origin and n is in the direction of the perpendicular *from* the origin *to* the line (or plane); p is negative if the line (or plane) does not pass through the origin and n has the opposite direction; p is zero if the line (or plane) passes through the origin. Conversely, if α, β (or α, β, γ) are direction cosines, this equation represents a line (or plane) at a distance p from the origin, whose normal has these direction cosines.