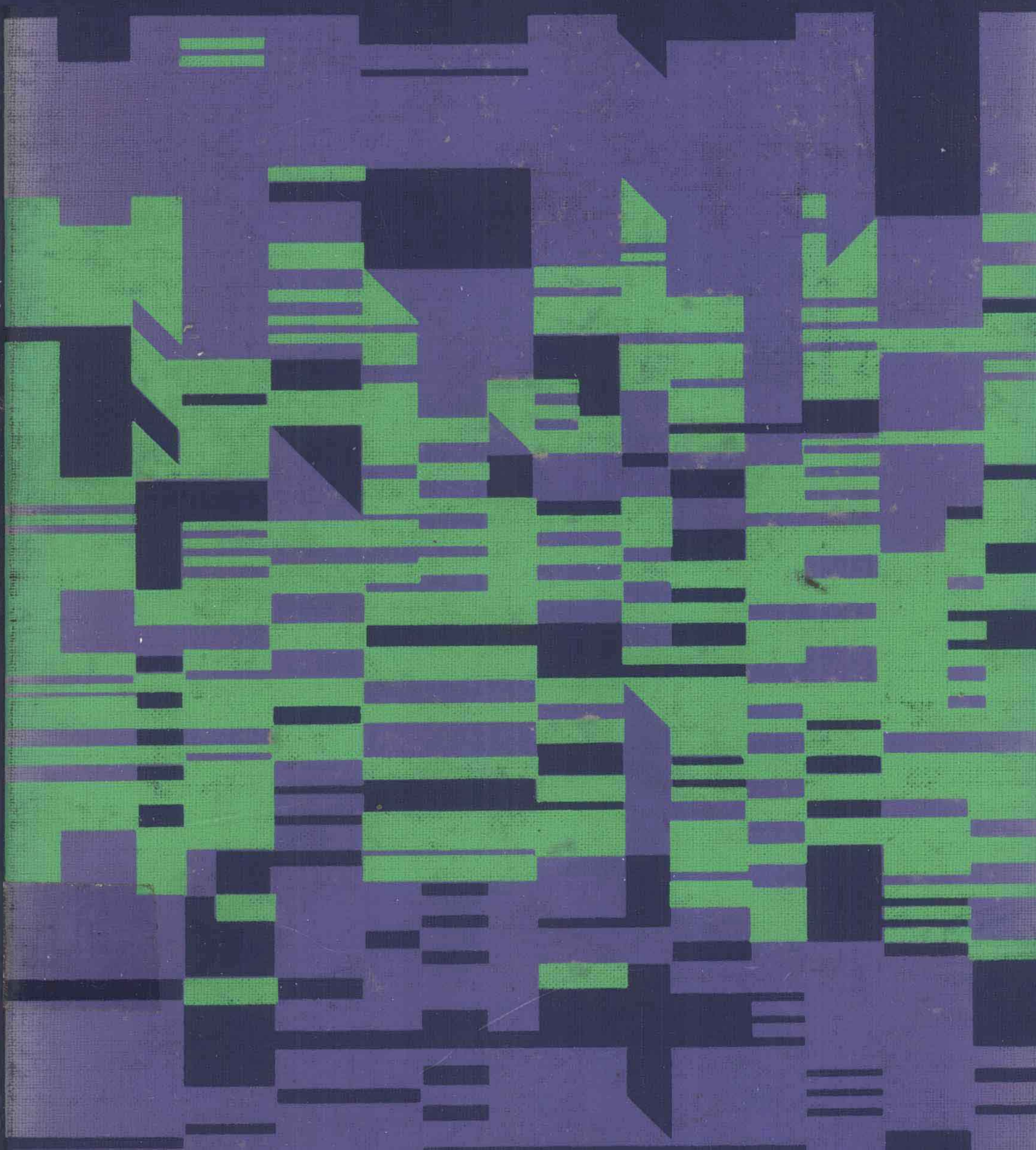
Introductory Algebra for College Students

Eugene Nichols



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PREFACE

This is an introductory algebra text. It contains all of the topics necessary for studying college algebra; they are developed in a way as to maintain proper balance between theory and application. The book is intended for those college students who have not had an algebra course in high school or who have had a strictly traditional high-school algebra course.

Some of the more prominent features of this textbook are the following:

1. The basic concepts of sets are introduced in the first chapter and used throughout the book. This approach provides an elegant way of expressing many mathematical relations and helps clarify and unify algebraic concepts. The first chapter should be used as a reference or review by those students who have already studied sets.
2. Throughout, there is emphasis on reasoning and proof; but the idea of proof and practice in writing proofs are introduced very gradually. The student is first taught to recognize valid proofs and to supply reasons for the statements in proofs.

3. A consistent effort has been made to provide exposition which is simple and clear so that the student is encouraged to read the textbook on his own.
4. An unusually large number of problems is supplied so that the instructor has a flexibility in deciding on the lengths of assignments.

A glossary of new terms is provided at the end of each chapter. The student should use it to review the vocabulary of the chapter. Chapter review problems are also provided at the end of each chapter. They can be used as review or as a test.

Eugene Nichols

Tallahassee, Florida
November 1970

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1

Sets

1.1 Membership, Finite Sets, Infinite Sets, and Equivalent Sets

Two mathematicians, George Boole (1815–1864) and Georg Cantor (1845–1918), are credited with the development of many ideas of sets. Cantor is considered to be the founder of set theory. In honor of Boole, who was the first to introduce some ideas of sets, the algebra of sets is frequently called *Boolean Algebra.*

A set is simply a collection of things. Some examples of sets are:

A: the set consisting of the numbers 1, 5, 10, 20
B: the set consisting of all students in this course
C: the set of all citizens of the United States

Now consider three other sets.

D: the set of all natural numbers; that is, 1, 2, 3, 4, . . .
E: the set of all even natural numbers; that is, 2, 4, 6, 8, . . .
F: the set of all odd natural numbers; that is, 1, 3, 5, 7, . . .

In sets D, E, and F, the three dots indicate that each sequence of numbers continues according to the pattern suggested by the listed numbers. It is easy to discover each pattern and continue listing subsequent numbers, if one should desire to do so.

Each object which belongs to a set is said to be a *member* or an *element* of the set. The following notation is used to say that 5 is an element of set A:

$$5 \in A,$$

which is read:

five is a member of A,

or

five is an element of A,

or

five belongs to A.

To say that 3 is not a member of A, the following notation is used:

$$3 \notin A.$$

This is read:

three does not belong to A.

There is one very essential difference between sets A, B, C, and D, E, F. Each of the sets A, B, and C is a *finite* set and each of the sets D, E, and F is an *infinite* set.

To simplify writing, braces are used to indicate a set. For example,

$$A = \{1, 5, 10, 20\}$$
$$D = \{1, 2, 3, \ldots\}.$$

In referring to infinite sets, it is not possible to list all of their elements. It is important to list enough elements to establish the pattern according to which the subsequent elements are arranged.

There is one important relation which exists between pairs of some sets. It can be illustrated by using a pair of finite sets.

$$X = \{1, 2, 3\} \qquad Y = \{100, 101, 102\}.$$

A *one-to-one correspondence* can be established between sets X and Y by matching each element of X with exactly one element of Y. This can be done in several ways. Three of these ways are the following:

$1 \leftrightarrow 100$		$1 \leftrightarrow 100$		$2 \leftrightarrow 100$
$3 \leftrightarrow 101$	or	$2 \leftrightarrow 101$	or	$1 \leftrightarrow 101$
$2 \leftrightarrow 102$		$3 \leftrightarrow 102$		$3 \leftrightarrow 102.$

It is possible to reason out the total number of different ways in which the sets X and Y can be matched. 1 in X can be matched in three different ways with an element in Y: with 100, with 101, or with 102. After that, 2 can be matched in two different ways and 3 in one way. Thus, the total number of matchings is $3 \times 2 \times 1$, or 6.

DEFINITION 1.1 Two sets between which there exists a one-to-one correspondence are said to be *equivalent* sets.

It is easy to conclude that two finite equivalent sets have the same number of elements. For example, each of the sets X and Y has three elements. The notation $n(X) = 3$ (read: n of X is three) will be used to say that the number of elements in set X is 3.

Consider next the matter of a one-to-one correspondence between two infinite sets. Let N be the set of natural numbers and E the set of even natural numbers.

$$N = \{1, 2, 3, \ldots\}$$
$$E = \{2, 4, 6, \ldots\}.$$

Since these sets are not finite, it is not possible to display the complete matching. To show that there is a one-to-one correspondence between two infinite sets, it is necessary to begin matching according to a pattern which will continue. For the sets N and E above, this pattern is indicated by the following:

$$\begin{array}{lllllllllll} N: & \{1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ E: & \{2, & 4, & 6, & 8, & 10, & 12, & 14, & 16, & \ldots\}. \end{array}$$

Each arrow above shows the two elements which are matched with each other. The pattern suggested by this matching is the following:

To each k in N there corresponds $2k$ in E.

To each t in E there corresponds $\frac{t}{2}$ in N.

Note that t is an even number, thus $\frac{t}{2}$ is a natural number.

Let us consider another pair of infinite sets.

$$\begin{array}{llllllllll} X: & \{1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \ldots\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ Y: & \{100, & 101, & 102, & 103, & 104, & 105, & 106, & 107, & \ldots\}. \end{array}$$

The set X is the set of all natural numbers and the set Y is the set of natural numbers which are greater than 99. Each set is infinite. In general, n in X is matched with $n + 99$ in Y and k in Y is matched with $k - 99$ in X.

For some exercises below it will be necessary to know the concept of divisibility. It is defined as follows:

DEFINITION 1.2 The natural number x is *divisible* by the natural number y if and only if $x \div y$ is a natural number; y is said to be a *divisor* of x.

Divisibility of natural numbers by 3 and by 9 is particularly interesting. It can be shown that a natural number is divisible by 3 if and only if the sum of its digit values is divisible by 3. For example, the sum of the digit values of 73,062 is $7 + 3 + 0 + 6 + 2 = 18$ and since 18 is divisible by 3, so is 73,062. Similarly, a natural number is divisible by 9 if and only if the sum of its digit values is divisible by 9. For example, the sum of the digit values of 1854 is divisible by 9, therefore 1854 is divisible by 9. Since the sum of the digit values of 265 is divisible by neither 3 nor 9, 265 is divisible by neither 3 nor 9.

It is easy to observe that if a number is divisible by 9, then it is also divisible by 3.

EXERCISE 1.1

1. Are the following pairs of sets matching sets?

a. S: {2, 3, 4, 5, 6}
T: {Mike, Bob, Hal, Leroy, Ed, Bill}

b. C: {Scranton, Florida, California}
D: {St. Paul, Chicago, Denver, Vermont}

c. G: {1, 3, 5, 7, 9, . . .}
H: {2, 4, 6, 8, 10}

2. Give examples of three pairs of *finite* sets such that the two sets in each pair are matching sets.

3. Give examples of three pairs of finite sets such that *no* two sets in a pair are matching sets.

4. Give examples of three pairs of *infinite* sets such that the two sets in each pair are matching sets.

5. For each pair of sets, a one-to-one correspondence is shown. Describe the one-to-one correspondence between the elements of A and those of B.

Example A: {1, 2, 3, 4, . . .}
↕ ↕ ↕ ↕
B: {1, 4, 9, 16, . . .}.

Description: Each element of A is matched with its square ($1^2 = 1$; $2^2 = 4$; $3^2 = 9$; and so on).

a. A: $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{2, 3, 4, 5, \ldots\}$

b. A: $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{3, 4, 5, 6, \ldots\}$

c. A: $\{\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{3, 5, 7, 9, \ldots\}$

d. A: $\{1, 2, 3, 4, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{1, 8, 27, 64, \ldots\}$

e. A: $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\}$

f. A: $\{1, 2, 3, 4, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{2, 5, 8, 11, \ldots\}$

g. A: $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{\frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \ldots\}$

h. A: $\{1, 2, 3, 4, 5, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{2, 3, 5, 7, 11, \ldots\}$

i. A: $\{1, 2, 3, 4, 5, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{2, 8, 18, 32, 50, \ldots\}$

j. A: $\{1, 2, 3, 4, 5, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{3, 9, 19, 33, 51, \ldots\}$

k. A: $\{1, 2, 3, 4, 5, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{3, 10, 29, 66, 127, \ldots\}$

l. A: $\{\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots\}$
$\updownarrow\ \updownarrow\ \updownarrow\ \updownarrow$
B: $\{3, 7, 11, 15, \ldots\}$

6. List each of the following sets.

a. the set of natural numbers which are less than 6

b. the set of months whose names start with the letter D

c. the set of natural numbers which are less than 9 and greater than 6

d. the set of odd natural numbers which are less than 20

e. the set of the planets of the solar system

f. the set of the seasons of the year

g. the set of odd natural numbers which are less than 20 and are divisible by 5

h. the set of natural numbers between 50 and 100 which are divisible by 6

7. What do the three sets described below have in common? [*Hint:* First list their elements.]

the natural numbers less than 7 which are also greater than 5

the odd natural numbers less than 20 which are also divisible by 7

the states of the United States entirely surrounded by water

8. Let T be the set of natural numbers divisible by 3. Label each statement with T for true and F for false.

a. $39 \in T$
b. $17 \notin T$
c. $117 \notin T$
d. $237 \in T$
e. $112 \in T$
f. $10{,}001 \in T$
g. $522 \notin T$
h. $92{,}322 \in T$
i. $92.321 \in T$

9. Let S be the set of natural numbers divisible by 9. Label each statement as either true or false.

a. $81 \notin S$
b. $89 \in S$
c. $207 \in S$
d. $88{,}092 \in S$
e. $34{,}560 \notin S$
f. $88{,}776 \in S$

10. Give an example of a number which is divisible by 3, but is not divisible by 9.

***11.** Prove the divisibility rule by 3.

***12.** Prove the divisibility rule by 9.

1.2 Numbers and Numerals

Consider the question, "how many numbers *five* are there?" The answer is: there is only *one* number five. Yet the number five has many names. When we want to refer to the number five, we may use any one of many names, such as

$$5 \qquad \frac{10}{2} \qquad 3+2 \qquad \text{V} \qquad \text{five} \qquad 2\tfrac{1}{2}\times 2.$$

A name of a number is called a *numeral*. In algebra, numbers are the subject of study most of the time. Occasionally, however, reference may be made to numerals. For example, the sentence

V is a Roman numeral for the number 5

is about a name of the number 5.

Sometimes it is a little awkward to be careful about distinguishing between numbers and numerals. For example, when saying "Give an example of a two-digit number divisible by 7," we are somewhat careless. It is because, when saying "a two-digit number," we really mean a two-digit numeral, since we tell the number of digits by looking at a symbol. For example, "43" has two digits, "4" and "3." To be quite careful, we would have to rephrase the statement above to read something like this:

Give an example of a two-digit numeral
which names a number divisible by 7.

At times in this book, we shall allow ourselves the luxury of avoiding the awkwardness due to the distinction between number and numeral.