QUASICONFORMAL MAPPINGS AND RIEMANN SURFACES

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FOREWORD

The theory of quasiconformal mappings is about half a century old. Its originators, Ahlfors, Grötzsch and Lavrent'yev approached quasiconformal mappings from three different points of view. Ahlfors, in working out the geometric meaning of the Nevanlinna theory (concerned with the value distribution of entire and meromorphic functions), made the important observation that his geometric version of the Nevanlinna theory does not require that the functions concerned be (locally) conformal mappings, but merely that they be (uniformly) quasiconformal. Lavrent'yev was looking for a geometric interpretation of the nonlinear partial differential equations of a two-dimensional steady potential flow of gas which would generalize the simple geometric meaning (namely, conformality) of the linear Cauchy-Riemann equations describing the two-dimensional potential flow of an incompressible fluid. Grötzsch, finally, posed and solved several extremal problems for nonconformal mappings. His great merit was to recognize that the "correct" way of measuring the deviation of a mapping from conformality is by the supremum of the local deviation rather than by some weighted average. This way is "correct" in the sense that it leads to beautiful and striking results.

The celebrated theorem by Teichmüller, obtained about ten years after Grötzsch's results should be considered as a far reaching deepening and extension of Grötzsch's beautiful but simple papers. The significance, and the very validity, of Teichmüller's theorem, was not recognized at once. This was due partly to the novelty of Teichmüller's ideas, partly to the unusual way in which they were presented, and perhaps also due to some other specific reasons. However, Ahlfors' 1950 paper in the Jerusalem Journal d'Analyse began a period of intensive assimilation and reworking of Teichmüller's approach. Only after this was accomplished did the theory of quasiconformal mappings enter the mainstream of mathematics.

Today, the theory of quasiconformal mappings in two dimensions is closely connected with the theory of moduli of Riemann surfaces, a connection originally established by Teichmüller, with uniformization theory and with the theory of Kleinian groups. Indeed, the methods suggested by quasiconformal mappings revived their venerable subjects and led to new interesting questions and results. Very recently new connections were established with topology in two and three dimensions, primarily through the spectacular discoveries by Thurston. On the other hand, the theory of quasiconformal self-mappings of the upper half-plane and of the so-called universal Teichmüller spaces and Kleinian groups are being actively studied by many investigators, and seem to attract the attention of mathematicians working in fields other than classical function theory. There are, however, few general expositions, and the English translation of S. L. Krushkal's book is a welcome addition to the literature. The book begins at the beginning, so that it may serve as an introduction to the subject. The bibliography is rather complete and will enable the reader to pursue further any subject mentioned. The western reader will be particularly grateful for the detailed report on the important work by the author himself and by other Soviet mathematicians on the variational problem by which Teichmüller initiated the modern theory and on various generalizations of this problem.

Lipman Bers
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PREFACE

Recent decades have seen an intensive development of areas in the theory of functions of a complex variable that touch on both the theory of quasiconformal mappings and the theory of Riemann surfaces. The application of quasiconformal mappings has not only provided a new tool for the investigation of problems in the theory of Riemann surfaces (and, in particular, classical problems that have not been solved) but has also shed light on the basic nature of the most important concepts of that theory. On the other hand, many problems in the theory of quasiconformal mappings receive a natural completion when we examine them on Riemann surfaces and not merely on plane regions.

The present monograph is devoted to a study of these questions. It deals primarily with the study of extremal problems for quasiconformal (and conformal) mappings and the development of variational methods for solving them, with the theory of spaces of Riemann surfaces and some generalizations of them that are connected with uniformization and the classical problem of moduli of Riemann surfaces, and with the solution of certain problems in the theory of (discontinuous) Kleinian groups of transformations.

Here, the methods of the theory of functions are interwoven with the ideas of functional analysis, topology, partial differential equations, algebraic geometry, and other branches of mathematics.

A distinctive feature of the exposition made in the book is the systematic application of the variational method. In the theory of quasiconformal mappings and its applications, different investigators (M. A. Lavrent'yev, L. V. Ahlfors, P. P. Belinskiy, and others) have used different variational methods, though these are akin to each other in a certain sense. The approach that we shall take rests largely on the methods of functional analysis.

In Chapter I, which is of an auxiliary nature, we shall give some general information regarding quasiconformal mappings, Riemann surfaces, and discontinuous transformation groups. The basic results are treated in the remaining six chapters. The reader who is primarily interested in the theory of conformal mappings can read Chapter IV independently of the others (once he has passed §2 of Chapter I).

I am deeply grateful to P. P. Belinskiy, with whom I have more than once discussed the questions examined here. B. N. Apanasov and V. V. Chuyeshev were very helpful in the preparation of the manuscript for printing. I am indebted to P. A. Biluta, L. I. Volkovskiy, and I. A. Volynets, who carefully read the manuscript, for a number of valuable comments. I express my gratitude to all.

Samuil L. Krushkal'

CONTENTS

Chapter 1

GENERAL INFORMATION

§1. Certain spaces of functions. Integral operators

1. Consider the following complex Banach spaces: $L_p(E)$, where $p \geqslant 1$, the space of functions $f(z)$ that are measurable on a set E in the plane of the complex variable $z = x + iy$ with norm defined by

$$\|f\|_p \equiv \|f\|_{L_p(E)} = \left(\iint\limits_E |f(z)|^p \, dx dy \right)^{1/p}; \tag{1}$$

$L_\infty(E)$, the space of functions $f(z)$ that are measurable on E with norm defined by

$$\|f\|_\infty \equiv \|f\|_{L_\infty(E)} = \sup_{z \in E} |f(z)|; \tag{2}$$

$C(F)$, the space of continuous functions $f(z)$ defined on a closed set F with norm defined by

$$\|f\|_{C(F)} = \max_{z \in F} |f(z)|; \tag{3}$$

$C_\alpha(F)$, the space of functions $f(z)$ that are defined on a closed set F and that satisfy a Hölder condition with exponent α in (0, 1] with norm

$$\|f\|_{C_\alpha(E)} = \|f\|_{C(F)} + H_{\alpha,F}(f), \tag{4}$$

where

$$H_{\alpha,F}(f) = \sup_{z_1, z_2 \in F} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha}; \tag{5}$$

$C_\alpha^m(\overline{D})$, where $C_\alpha^0 \equiv C_\alpha$, the space of functions $f(z)$ that have continuous derivatives of the first m orders in a closed region $\overline{D}$ such that

$$\frac{\partial^m f}{\partial x^{m-l} \partial y^l} \in C_\alpha(\overline{D}) \quad (l = 0, 1, \ldots, m), \ 0 < \alpha < 1, \ m \geqslant 0,$$

where the derivatives at a boundary point z are defined as the limits of the corresponding derivatives as $z \to z_0$ through values inside D and the norm is defined as

$$\|f\|_{C_\alpha^m(\overline{D})} = \sum_{k=0}^{m} \sum_{l=0}^{k} \left\| \frac{\partial^k f}{\partial x^{k-l} \partial y^l} \right\|_{C(\overline{D})} + \sum_{l=0}^{m} H_{\alpha,\overline{D}} \left(\frac{\partial^m f}{\partial x^{m-l} \partial y^l} \right); \tag{6}$$

$B_{p,R}$ (where $p > 2$), the space of functions $f(z)$, with $f(0) = 0$, that are defined on the disk $\overline{U}_R$: $|z| \leqslant R$ (where $0 < R < \infty$) with norm

$$\|f\|_{B_{p,R}} = H_{1-\frac{2}{p}, \overline{U}_R}(f) + \|f_z\|_p + \|f_{\bar{z}}\|_p, \tag{7}$$

where, as usual,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{8}$$

Here and in the rest of the book, the derivatives are understood to be generalized derivatives in the sense of Sobolev, that is, in the sense of the theory of generalized functions.

We shall denote the disk $|z| < R$ by U_R. For $R=1$, we shall denote the unit disk $|z| < 1$ simply by U instead of U_1. We note also that in formula (7) we can take $R=\infty$, assuming that (5), (7), and (8) are considered only for finite z.

By virtue of the familiar embedding theorems of Sobolev [164], we have

$$H_{1-\frac{2}{p}, \bar{U}_R}(f) \leqslant M(p, R)\left(\|f_z\|_p + \|f_{\bar{z}}\|_p\right), \quad 0 < R < \infty. \tag{9}$$

Here, the constant $M(p, R)$ depends only on p and R. It follows from (3), (5), and (7) that

$$\|f\|_{C(\bar{U}_R)} \leqslant R^{(p-2)/p}\|f\|_{B_{p,R}}, \quad 0 < R < \infty. \tag{10}$$

2. Let $\mathbf{C}$ denote the complex plane. In the space $L_p(\mathbf{C})$, where $p > 2$, let us look at the integral operators

$$T_0\rho(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \rho(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta}\right) d\xi d\eta, \tag{11}$$

$$\Pi\rho(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{\rho(\zeta) - \rho(z)}{(\zeta - z)^2}\, d\xi d\eta, \quad \zeta = \xi + i\eta. \tag{12}$$

The integral (11) converges absolutely. By virtue of the results of Calderón and Zygmund [79], the integral (12) exists for almost all $z \in \mathbf{C}$ as a Cauchy principal value. If $\rho(z)$ has a compact support, then

$$\Pi\rho = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{\rho(\zeta)\,d\xi d\eta}{(\zeta - z)^2}. \tag{12'}$$

Theorem 1. *If* $\rho \in L_p(\mathbf{C})$ *for all* $p > 2$, *then*

$$(T_0\rho)_{\bar{z}} = \rho, \quad (T_0\rho)_z = \Pi\rho, \tag{13}$$

$T_0\rho \in B_{p,\infty}$, *and*

$$\|T_0\rho\|_{B_{p,\infty}} \leqslant M_1(p)\,\|\rho\|_p, \tag{14}$$

where the constant $M_1(p)$ *depends only on* p *and* $\Pi\rho$ *is a (bounded) linear operator from* $L_p(\mathbf{C})$ *into* $L_p(\mathbf{C})$ *such that*

$$\|\Pi\rho\|_p \leqslant \Lambda_p\,\|\rho\|_p, \tag{15}$$

where $\Lambda_p \equiv \|\Pi\|_{L_p}$ *(the norm of the operator* Π *in* $L_p(\mathbf{C})$*) depends continuously on* p *and* $\Lambda_2 = 1$.

It follows that, for arbitrary k in $(0, 1)$, there exists a $\delta = \delta(k) > 0$ such that

$$k\Lambda_p < 1 \quad \text{for} \quad 2 < p < 2 + \delta = p_0(k). \tag{16}$$

If $\rho(z)$ belongs to C_α^m and has a compact support, (13) will have derivatives in the usual sense, $T_0\rho$ will belong to C_α^{m+1} and $\Pi\rho$ will belong to C_α^m.

In what follows, we shall also apply the operator

$$T\rho(z) = -\frac{1}{\pi}\iint_{\mathbf{C}} \frac{\rho(\zeta)\,d\xi d\eta}{\zeta - z} \tag{17}$$

to functions of compact support $\rho \in L_p(\mathbf{C})$, where $p > 2$. Obviously, $T_0\rho(z) = T\rho(z) - T\rho(0)$. For the operator $T\rho$, we have relationships analogous to (13) and (14) (except that, in general,

$T\rho(0) \neq 0)$. If $\rho \in L_\infty(\mathbf{C})$, we have

$$|T\rho(z_1) - T\rho(z_2)| \leqslant M_2 \, \|\rho\|_\infty \, |z_1 - z_2| \, |\ln |z_1 - z_2||, \tag{18}$$

where z_1 and z_2 are complex numbers and M_2 is a positive constant.

Proofs of these assertions can be found, for example, in the books by Vekua [43], Lehto and Virtanen [126], and Ahlfors [9] (see also [79]). That Λ_p depends continuously on p follows from Riesz's theorem [160] on the logarithmic convexity of Λ_p^p as a function of p.

We note also that for functions $f(z)$ that are continuous in a closed region $D \subset \mathbf{C}$ of finite connectivity with rectifiable boundary ∂D and that have generalized derivatives $f_{\bar{z}}$ belonging to $L_p(D)$, where $p > 2$, Green's formula

$$\iint_D \frac{\partial f}{\partial \bar{z}} \, dx dy = \frac{1}{2i} \int_{\partial D} f(z) \, dz \tag{19}$$

and the Borel-Pompeiu formula

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta) \, d\zeta}{\zeta - z} - \frac{1}{\pi} \iint_D \frac{\partial f}{\partial \bar{\zeta}} \frac{d\xi d\eta}{\zeta - z} = \begin{cases} f(z), & z \in D, \\ 0, & z \in C \setminus \overline{D} \end{cases} \tag{20}$$

remain valid. These formulas are easily obtained, for example, by smoothing $f(z)$ and applying the corresponding formulas for smooth functions.

It follows from (20) that, *if the generalized derivative $f_{\bar{z}}$ vanishes in the region D, then $f(z)$ is holomorphic in D.*

§2. Quasiconformal mappings of plane regions

1. Suppose that a function $w = f(z) = u(x, y) + iv(x, y)$ maps a region D contained in the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$

homeomorphically onto a region Δ and that it has locally square-integrable generalized derivatives f_z and $f_{\bar{z}}$ in D. It then follows in particular that $f(z)$ is differentiable almost everywhere in D (see, for example, [139, 158]). For definiteness, we shall treat only orientation-preserving mappings.

Following Lavrent'yev [115, 117], we shall refer to the real numbers $p=p(z_0) \geqslant 1$ and $\theta=\theta(z_0)$ as the *characteristics* of the mapping $w=f(z)$ at the point $z_0 \in D$ if an infinitesimally small ellipse E_0 with center at the point z_0, with semiaxis ratio $a/b=p$ (where $a \geqslant b$), and with the major semiaxis inclined at an angle θ to the x-axis, is transformed in such a way that

$$\lim_{a \to 0} \frac{\max\limits_{z \in E_0} |f(z) - f(z_0)|}{\min\limits_{z \in E_0} |f(z) - f(z_0|} = 1 . \tag{21}$$

(If $a=b$, θ is undefined.) Suppose that a measurable function $p(z) \geqslant 1$ is defined on a region D and that a measurable function $\theta(z)$ is defined on that part of D on which $p(z) > 1$. Suppose that (21) holds for these functions almost everywhere in D and that $p(z) \in L_\infty(D)$. Then, we shall call the homeomorphism f a *quasiconformal mapping* with *characteristics* $p(z)$ and $\theta(z)$.

At points of differentiability of f, the characteristics are connected with the derivatives f_z and $f_{\bar{z}}$ by

$$-\frac{p(z)-1}{p(z)+1} e^{2i\theta(z)} = \frac{f_{\bar{z}}(z)}{f_z(z)}.$$

Therefore, we can give the following definition of quasiconformality, which is equivalent to the preceding one. We define a quasiconformal mapping of the region D as any homeomorphic generalized solution $w=f(z)$ of Beltrami's equation

$$w_{\bar{z}} - \mu(z) w_z = 0, \tag{22}$$

where $\mu(z)$ is a measurable function in D that satisfies the condition $\|\mu\|_\infty < 1$, that is, a solution such that the derivatives f_z and $f_{\bar{z}}$ are locally square-integrable generalized derivatives and $f(z)$ satisfies equation (22) almost everywhere in D. This approach to a quasiconformal mapping is due to Vekua [43, 44] (see also Morrey [144]).

The mapping f is conformal in the Riemannian metric

$$ds^2=\lambda(z)\ |dz+\mu(z)d\bar{z}|^2,\ \ \lambda(z) > 0 \tag{23}$$

and it preserves angles almost everywhere if these are measured in the region D in terms of the metric (23) but in the region $\Delta=f(D)$ in terms of the usual Euclidean metric $ds_1^2 = du^2 + dv^2$.

In particular, *if* $\mu(z)=0$ *almost everywhere in the region D, then f(z) is a holomorphic function in D.*

Henceforth, *when we speak of solutions of equation* (22), *we shall always mean generalized solutions.*

The function

$$\mu_f\ (z) = \frac{f_{\bar{z}}}{f_z} \tag{24}$$

is called the *complex dilatation* or *Beltrami coefficient* of the mapping f, and the ratio

$$K\ (f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty} \tag{25}$$

is called the *dilatation* of the mapping f. From (24) and (25), we have

$$K\ (f) = \sup_{z\in D} \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}.$$

A mapping f such that $K(f)=K$ is said to be *K-quasiconformal.* K-quasiconformality is equivalent to satisfaction of the inequality

$$|f_z|^2 + |f_{\bar z}|^2 \leqslant \frac{1}{2}\left(K + \frac{1}{K}\right)\left(|f_z|^2 - |f_{\bar z}|^2\right). \qquad (26)$$

If the homeomorphism f is not quasiconformal, then $K(f) = \infty$.

A very simple example of a quasiconformal mapping is the affine mapping $w = az + b\bar z + c$, where a, b, and c are constants. Its complex characteristic is

$$\frac{w_{\bar z}}{w_z} = \frac{b}{a} = \text{const},$$

and it maps every circle into an ellipse.

Let $w = f(z)$ denote any quasiconformal mapping. In a neighborhood of a point z_0 at which this mapping is differentiable, we have

$$w - w_0 =$$

$$\frac{\partial f(z_0)}{\partial z}(z - z_0) + \frac{\partial f(z_0)}{\partial \bar z}(\bar z - \bar z_0) + o(|z - z_0|), \; w_0 = f(z_0),$$

so that, at points where it is differentiable, every quasiconformal mapping behaves in the small like an affine mapping.

2. Let us formulate the general properties of quasiconformal mappings that we shall need later. Proofs of the assertions that we shall make can be found, for example, in the books by Ahlfors [9], Belinskiy [25], Vekua [43], and Lehto and Virtanen [126].

Theorem 2. *For an arbitrary measurable complex dilatation*

$$\mu(z) = -\frac{p(z) - 1}{p(z) + 1} e^{2i\theta(z)}, \quad \|\mu\|_\infty < 1, \qquad (27)$$

defined in a simply-connected region D contained in $\overline{\mathbf{C}}$, there exists a homeomorphism $w = f(z)$ of the region D onto a given simply-connected region Δ (of the same conformal type as D) that satisfies Beltrami's equation $w_{\bar z} = \mu(z) w_z$.

Theorem 3. *Let $f_0(z)$ denote a homeomorphic solution of*