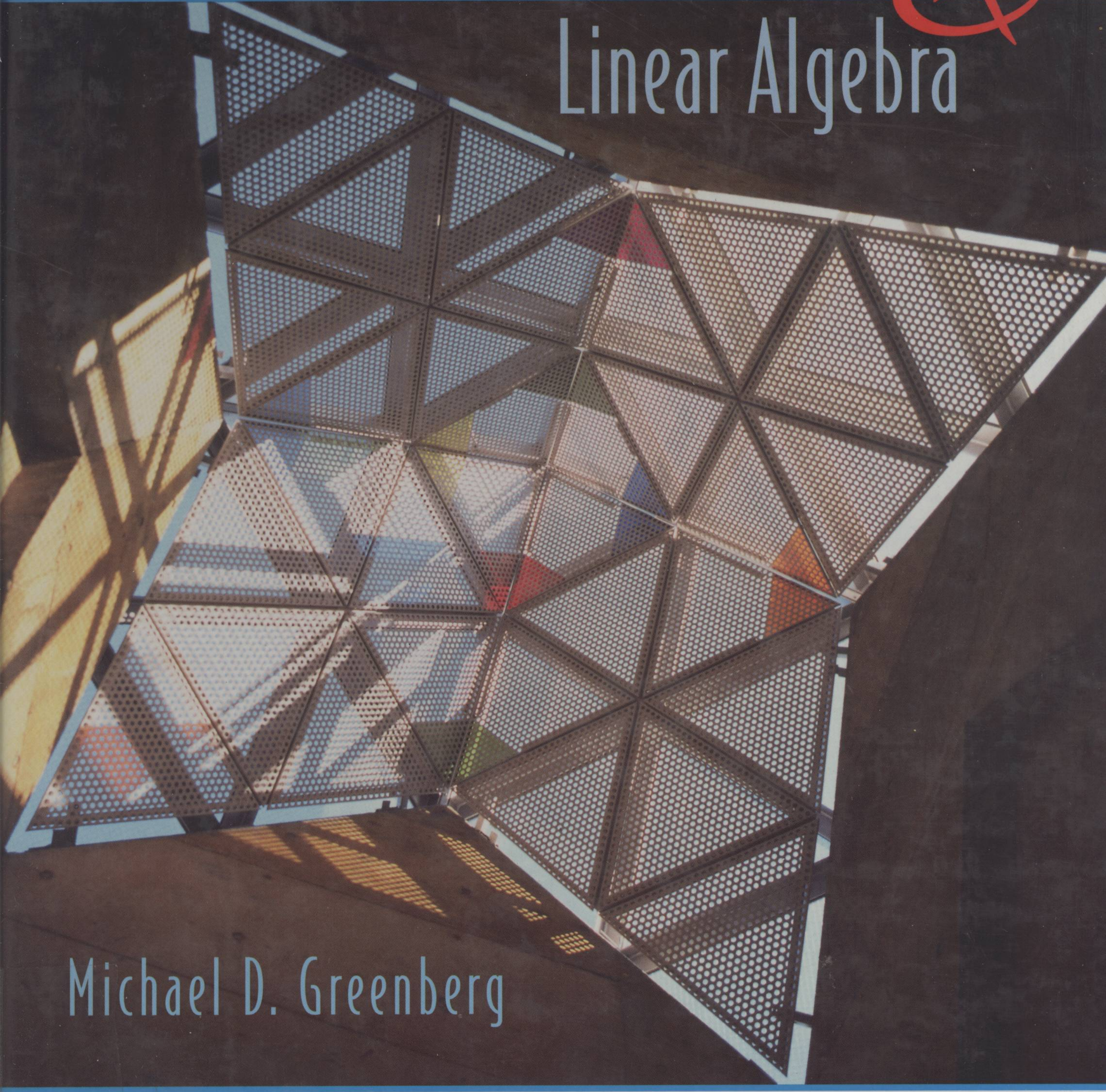
Differential Equations & Linear Algebra



Michael D. Greenberg

Differential Equations & Linear Algebra

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Preface

PURPOSE AND PREREQUISITES

This book is intended as a textbook for a course in differential equations with linear algebra, to follow the differential and integral calculus. Since the syllabus of such a course is by no means standard, we have included more material than can be covered in a single course—possibly enough material for a two-semester course. This additional material is included to broaden the menu for the instructor and to increase the text's subsequent usefulness as a reference book for the student.

Written for engineering, science, and computer science students, the approach is aimed at the applications oriented student but is also intended to be rigorous and to reveal the beauty and elegance of the subject.

Why blend linear algebra with the differential equations? Since mid-twentieth century, the traditional course in differential equations has been offered in the first or second semester of the sophomore year and has relied on only a minimum of linear algebra, most notably the use of determinants. More recently, beginning with the advent of digital computers on campuses and in industry around the 1960s, a course or part of a course in linear algebra has become a part of most engineering science curricula. Given the current interest in introducing linear algebra earlier in curricula, the growing importance of systems of differential equations, and the natural use of linear algebra concepts in the study of differential equations, it seems best to move toward an integrated approach.

FLEXIBILITY

The text is organized so as to be flexible. For instance, it is generally considered desirable to include some nonlinear phase plane analysis in a course on differential equations since the qualitative topological approach complements the traditional analytical approach and also powerfully emphasizes the differences between linear and nonlinear systems. However, that topic usually proves to be a "luxury" to which one can devote one or two classes at best. Thus, we have arranged the phase plane material to allow anywhere from a one-class introduction to a moderately detailed discussion: We introduce the phase plane in only four pages in Section 7.3 in support of our discussion of the harmonic oscillator and we return to it in Chapter 11. There, Section 11.2 affords a more detailed overview of the method and provides another possible stopping point.

To assist the instructor in the syllabus design we list some sections and subsections as optional but emphasize that these designations are subjective and intended only as a rule of thumb. (To the student we note that "optional" is not intended to mean unimportant, but only as a guide as to which material can be omitted by virtue of not being a prerequisite for the material that follows.)

SPECIFIC PEDAGOGICAL DECISIONS

Several pedagogical decisions made in writing this text deserve explanation.

1. *Chapter sequence*: Some instructors prefer to discuss numerical solution early, even within the study of first-order equations. Placement of the material on numerical solution near the end of this text does not rule out such an approach for one could cover Sections 12.1-12.2 on Euler's method, say, at any point in Chapters 2 or 3. Here, it seemed preferable to group Chapters 11 (on the phase plane) and 12 (on numerical solution) together since they complement the analytical approach, the former being qualitative and the latter being quantitative. As such, these two chapters might well have been made the final chapters, with the Laplace transform chapter moving up to precede or to follow Chapter 8 on series solution. Such movement is possible in a course syllabus since other chapters do not depend on series solution or on the Laplace transform. Also along these lines, it might seem awkward that Chapters 4 and 5 on vectors and matrices are separated from Chapter 9 on the eigenvalue problem. This separation may not be as great as it appears since in a one-semester course Chapter 8 might well be omitted. In any case in a combined approach to differential equations and linear algebra it seems logical to intersperse these two topics as naturally as possible rather than presenting them end-to-end. It may even be true that for optimal student retention it is good to have a gap between first meeting the linear algebra in Chapters 4 and 5 and returning to it in Chapter 9, so that it feels more like one is studying the subject twice.

2. *Placing Gauss elimination in Chapter 4 on vectors rather than in Chapter 5 on matrices and linear algebraic equations:* Just as one studies the real number axis before studying functions (mappings from one such axis to another), it seems appropriate to study vector spaces before studying matrices (which provide mappings from one vector space to another). In that case we find—in discussing span, linear dependence, bases, and expansions in Chapter 4—that we need to solve systems of coupled linear algebraic equations. Hence, we devote Section 4.5, which precedes that discussion, to Gauss elimination.

3. *Introducing the Heaviside function in the chapter on first-order differential equations rather than in the chapter on the Laplace transform*: If the forcing function is given piecewise, solution of the differential equation by a computer algebra system (*Maple* in this text) requires us to give a single expression for that function, and that can be accomplished using the Heaviside function. Further, including the Heaviside function in Chapter 2 makes it possible to include that topic even if the chapter on the Laplace transform is not covered.

Computer Algebra System

As a representative computer algebra system this text uses *Maple*, but does not assume prior knowledge of that system. The *Maple* discussion is confined to subsections at the end of most sections, immediately preceding the exercises; see, for example, Sections 2.2 and 2.3. The reader can bypass those discussions entirely since they are supplemental and intended to show the student how to carry out various *Maple* calculations relevant to the material in that section. In some cases they explain how text figures were generated. The view represented here is that it would be foolish not to use the powerful computer algebra systems that are now available, but that primary emphasis should continue to rest firmly on fundamentals and understanding of the theory and methods. See also the section on supplements, below.

EXERCISES

End-of-section exercises are of different kinds and are arranged, typically, as follows. First, and usually near the beginning of the exercise group, are exercises that follow up on small gaps in the reading, thus engaging the student more fully in the reading (e.g., Exercises 1 and 2 of Section 3.5). Second, there are usually numerous "drill" type exercises that ask the student to mimic steps or calculations that are essentially similar to those demonstrated in the text (e.g., there are 19 matrices to invert by hand in Exercise 1 of Section 5.6). Third, there are exercises that call for the use of *Maple* (e.g., Exercise 3 of Section 5.6 and Exercise 4 of Section 10.4). Fourth, some exercises involve physical applications (e.g., Exercise 22 of Section 2.4 on the distribution of a pollutant in a river, Exercises 17 and 18 of Section 5.6 on electrical circuits, and Exercise 14 of Section 5.8 on computer graphics). And, fifth, there are exercises intended to extend the text and increase its value as a reference book (e.g., Exercises 7-12 of Section 2.3 on the Bernoulli, Riccati, Alembert-Lagrange, and Clairaut equations, and Exercise 2 of Section 3.3 on envelopes). Answers to selected exercises (which are denoted in the text by underlining the equation number) are given at the end of the book.

SYLLABUS DESIGN

Designing a two-semester course is simple in the sense that one would probably cover virtually everything in the text. Thus, let us restrict our comments to the design of a one-semester course. As a general comment we note that sections and subsections are arranged with an eye toward flexibility. In Chapter 10, for instance, one could limit the coverage to Sections 10.1–10.3 or one could cover Sections 10.1, 10.2, and 10.4. As a specific example, at the University of Delaware mechanical engineers are currently required to take a three-course sequence in their sophomore year as follows. In the fall they take a three-credit course on differential equations and linear algebra following a syllabus somewhat as follows: Chapters 1–7 and 9–10 with these sections omitted—2.3.3, 2.4.2, 3.4, 4.4.2, 4.4.3, 4.5.6, 4.5.7, 4.8.3, 4.9.4, 4.9.5, 5.6.5, 5.7.2, 5.8, 6.6.2, 6.7.3, 6.7.4, 9.4.2, 9.4.3, 10.3.3, and 10.5–10.7.

In the Spring they take two more courses, one covering Laplace transforms, field theory, and partial differential equations, and the other covering numerical methods, including the numerical solution of ordinary and partial differential equations.

SUPPLEMENTS

For information regarding the Instructor's Solution Manual and other supplements, see the publisher's website, available 1/1/01 at www.prenhall.com/greenberg. The site will contain quizzes and other text related activities that will be free to all text users. Suggestions, comments, and errata will be gratefully received at the author's e-mail address given below.

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I dedicate this book, with love, to my wife Yisraela. Finally and most important: "From whence cometh my help? My help cometh from the Lord, who made heaven and earth." (Psalm 121).

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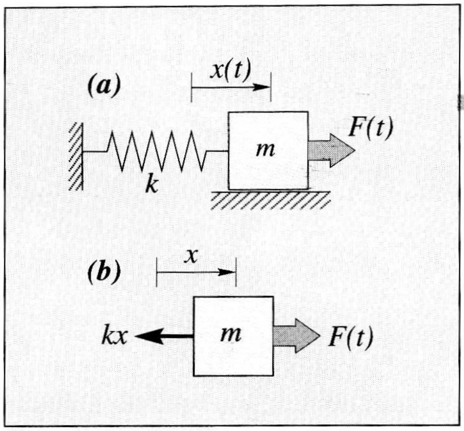

Chapter 1

INTRODUCTION TO DIFFERENTIAL EQUATIONS

1.1 INTRODUCTION

Most phenomena in science and engineering are governed by equations involving derivatives of one or more unknown functions.

To illustrate, consider the motion of a body of mass m along a straight line, which we designate as an x axis. Let the mass be subjected to a force $F(t)$ along that axis, where t is the time. Then according to Newton's second law of motion,

$$m\frac{d^2x}{dt^2} = F(t), \tag{1}$$

where $x(t)$ is the mass's displacement measured from the origin. If we know the displacement $x(t)$ and we wish to determine the force $F(t)$ required to produce that displacement, then the solution is simple: According to (1), we merely differentiate the given $x(t)$ twice and multiply the result by m.

However, if we know the applied force $F(t)$ and wish to determine the displacement $x(t)$ that results, then we say that (1) is a "differential equation" on $x(t)$ since it involves the derivative, more precisely the second derivative in this example, of the unknown function $x(t)$ with respect to t. The question is: What function $x(t)$, when differentiated twice with respect to t and then multiplied by m (which is a constant), gives the prescribed function $F(t)$? To solve (1) for $x(t)$, we need to undo the differentiations; that is, we need to integrate (1), twice in fact.

For definiteness and simplicity, suppose that $F(t) = F_0$ is a constant, so that (1) becomes

$$m\frac{d^2x}{dt^2} = F_0. \tag{2}$$

Integrating (2) once with respect to t gives

$$m\frac{dx}{dt} = F_0 t + A, \tag{3}$$

where A is an arbitrary constant of integration, and integrating again gives

$$mx = \frac{F_0}{2}t^2 + At + B,$$

so

$$x(t) = \frac{1}{m}\left(\frac{F_0}{2}t^2 + At + B\right). \tag{4}$$

The constants of integration, A and B, can be found from (3) and (4) if the displacement and the velocity are prescribed at the initial time, which we take to be $t = 0$. That is, we can solve for A and B if $x(0)$ and $\frac{dx}{dt}(0)$ are known. For instance, suppose we know that

$$x(0) = 0 \quad \text{and that} \quad \frac{dx}{dt}(0) = 0. \tag{5}$$

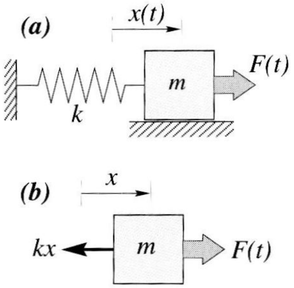

FIGURE 1
Mass/spring system.

Then, by setting $t = 0$ in (4), we find that $B = 0$, and by setting $t = 0$ in (3) we find that $A = 0$. Thus, (4) gives the solution

$$x(t) = \frac{1}{m}\frac{F_0}{2}t^2, \tag{6}$$

which evidently holds for all $t \geq 0$, that is, for $0 \leq t < \infty$. It is easily verified that (6) does satisfy both the differential equation (2) and the initial conditions (5).

Unfortunately, most differential equations cannot be solved this readily, that is, by merely undoing the derivatives by integration. For instance, suppose that the mass is restrained by a coil spring that supplies a restoring force (i.e., in the direction opposite to the displacement) proportional to the displacement x, with constant of proportionality k (Fig. 1a). Then (Fig. 1b) the total force on the mass is $-kx + F(t)$, so in place of (1), the differential equation governing the motion is

$$m\frac{d^2x}{dt^2} = -kx + F(t)$$

or,

$$m\frac{d^2x}{dt^2} + kx = F(t). \tag{7}$$

Let us try to solve (7) for $x(t)$ as we did before, by integrating twice with respect to t. After one integration (7) becomes

$$m\frac{dx}{dt} + k\int x(t)\,dt = \int F(t)\,dt + A, \tag{8}$$

where A is the constant of integration. Since $F(t)$ is a known function, the integral of $F(t)$ in (8) can be evaluated. However, since the solution $x(t)$ is not yet known, the integral $\int x(t)\,dt$ cannot be evaluated, and we cannot proceed with our intended technique of solution-by-repeated-integration. Be clear that it would be incorrect to state that the integral $\int x(t)\,dt$ in (8) is $x(t)t$. The latter would be correct if $x(t)$ were a constant, but $x(t)$ is a thus-far-unknown function that is probably not merely a constant. If we were to distinguish small mistakes from large ones, writing "$\int x(t)\,dt = x\,t$" would be a large one, so be sure to understand this point.

Thus, we see that solving differential equations is not merely a matter of undoing the derivatives by direct integration. The theory and technique involved is considerable and will occupy us throughout this book.

In order to develop the theory, it is convenient, and even necessary, to use concepts and methods from the mathematical domain known as Linear Algebra. Our plan is to get started in our study of differential equations until linear algebra concepts are about to force their way upon the discussion. At that point we take a "break," in Chapters 4 and 5, to consider as much linear algebra as needed to proceed with our discussion of differential equations. Additional linear algebra material, on the so-called "eigenvalue problem," is introduced in Chapters 9 and 10.

1.2 DEFINITIONS AND TERMINOLOGY

To begin, we introduce several fundamental definitions.

Differential equation. By a **differential equation** we mean an equation containing one or more derivatives of the function under consideration. Some examples, which we put forward without derivation, are as follows:

$$\frac{dN}{dt} = \kappa N, \tag{1}$$

$$m\frac{d^2x}{dt^2} + kx = F(t), \tag{2}$$

$$L\frac{d^2i}{dt^2} + \frac{1}{C}i = \frac{dE}{dt}, \tag{3}$$

$$\frac{d^2y}{dx^2} = C\sqrt{1+\left(\frac{dy}{dx}\right)^2}, \tag{4}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0, \tag{5}$$

$$EI\frac{d^4y}{dx^4} = -w(x). \tag{6}$$

Equation (1) is the differential equation governing the population $N(t)$ of a particular species, where κ is a net birth/death rate and t is the time.

Equation (2) governs the linear displacement $x(t)$ of a body of mass m, subjected to an applied force $F(t)$, and a restraining spring of stiffness k, as was discussed in Section 1.1.

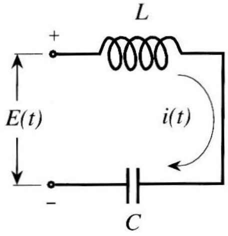

FIGURE 1
Electrical circuit, equation (3)

Equation (3) governs the current $i(t)$ in an electrical circuit containing an inductor with inductance L, a capacitor with capacitance C, and an applied voltage source of strength $E(t)$ (Fig. 1), where t is the time.

Equation (4) governs the deflection $y(x)$ of a flexible cable or string, hanging under the action of gravity, where C is a constant (Fig. 2).

Equation (5) governs the angular motion $\theta(t)$ of a pendulum of length l under the action of gravity, where g is the acceleration of gravity and t is the time (Fig. 3).

Finally, equation (6) governs the deflection $y(x)$ of a beam subjected to a load distribution $w(x)$ (Fig. 4), where E and I are physical constants that involve the beam material and cross sectional shape, respectively.

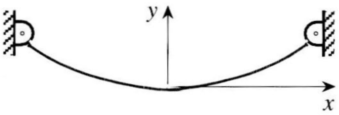

FIGURE 2
Hanging cable, equation (4).

Ordinary and partial differential equations. We classify a differential equation as an **ordinary differential equation** if it contains ordinary derivatives with respect to a single independent variable. Thus, equations (1)–(6) are ordinary differential equations (traditionally abbreviated as **ODE**'s). The independent variable is t in (1), (2), (3), and (5), and x in (4) and (6), but the mathematics that we develop will be insensitive to the physical nature of the independent and dependent variables.

If the dependent variable is a function of more than one independent variable then we can expect the governing differential equation to contain derivatives with respect to those various independent variables, partial derivatives this time, in which case we call the differential equation a **partial differential equation** (abbreviated as **PDE**). To illustrate, three of the most important PDE's in science and engineering are these:

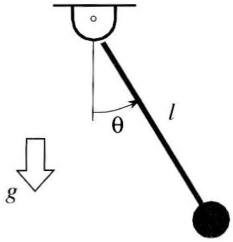

FIGURE 3
Pendulum, equation (5).

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=\rho c\frac{\partial T}{\partial t}, \tag{7}$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0, \tag{8}$$

$$\tau\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\rho\frac{\partial^2 w}{\partial t^2}, \tag{9}$$

which are examples of the **diffusion equation**, the **Laplace equation**, and the **wave equation**, respectively. The first governs the temperature distribution $T(x, y, z, t)$ in some domain of three-dimensional x, y, z space, such as the interior of a hot ingot that is cooling down under the action of the heat transfer mechanism known as conduction; the physical constants k, ρ, c are the conductivity, mass density, and specific heat of the medium, respectively. In physical terms, (7) expresses the thermodynamic law that the rate of change of the heat contained in any volume element of the material (given by the right-hand side of the equation) is equal to the net rate of heat flowing into the element through its surface by the mechanism of conduction (given by the left-hand side of the equation). The second, equation (8), governs the same phenomenon but in the event that the temperature distribution is in "steady state"—that is, it is not changing with time. And the third, equation (9), governs the deflection $w(x, y, t)$ of a stretched membrane such as a drumhead, normal to the x, y

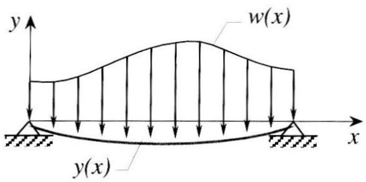

FIGURE 4
Loaded beam, equation (6).