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Aslak Tveito
Ragnar Winther

TEXTS IN APPLIED MATHEMATICS

Introduction to Partial Differential Equations

A Computational Approach

Springer

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Introduction to Partial Differential Equations

A Computational Approach

With 69 Illustrations

Springer

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Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics* (*TAM*).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences* (*AMS*) series, which will focus on advanced textbooks and research-level monographs.

Preface

"It is impossible to exaggerate the extent to which modern applied mathematics has been shaped and fueled by the general availability of fast computers with large memories. Their impact on mathematics, both applied and pure, is comparable to the role of the telescopes in astronomy and microscopes in biology."

— *Peter Lax, Siam Rev. Vol. 31 No. 4*

Congratulations! You have chosen to study partial differential equations. That decision is a wise one; the laws of nature are written in the language of partial differential equations. Therefore, these equations arise as models in virtually all branches of science and technology. Our goal in this book is to help you to understand what this vast subject is about. The book is an introduction to the field. We assume only that you are familiar with basic calculus and elementary linear algebra. Some experience with ordinary differential equations would also be an advantage.

Introductory courses in partial differential equations are given all over the world in various forms. The traditional approach to the subject is to introduce a number of analytical techniques, enabling the student to derive exact solutions of some simplified problems. Students who learn about computational techniques on other courses subsequently realize the scope of partial differential equations beyond paper and pencil.

Our approach is different. We introduce analytical and computational techniques in the same book and thus in the same course. The main reason for doing this is that the computer, developed to assist scientists in solving partial differential equations, has become commonly available and is currently used in all practical applications of partial differential equations. Therefore, a modern introduction to this topic must focus on methods suitable for computers. But these methods often rely on deep analytical insight into the equations. We must therefore take great care not to throw away basic analytical methods but seek a sound balance between analytical and computational techniques.

One advantage of introducing computational techniques is that nonlinear problems can be given more attention than is common in a purely analytical introduction. We have included several examples of nonlinear equations in addition to the standard linear models which are present in any introductory text. In particular we have included a discussion of reaction-diffusion equations. The reason for this is their widespread application as important models in various scientific applications.

Our aim is not to discuss the merits of different numerical techniques. There are a huge number of papers in scientific journals comparing different methods to solve various problems. We do not want to include such discussions. Our aim is to demonstrate that computational techniques are simple to use and often give very nice results, not to show that even better results can be obtained if slightly different methods are used. We touch briefly upon some such discussion, but not in any major way, since this really belongs to the field of numerical analysis and should be taught in separate courses. Having said this, we always try to use the simplest possible numerical techniques. This should in no way be interpreted as an attempt to advocate certain methods as opposed to others; they are merely chosen for their simplicity.

Simplicity is also our reason for choosing to present exclusively finite difference techniques. The entire text could just as well be based on finite element techniques, which definitely have greater potential from an application point of view but are slightly harder to understand than their finite difference counterparts.

We have attempted to present the material at an easy pace, explaining carefully both the ideas and details of the derivations. This is particularly the case in the first chapters but subsequently less details are included and some steps are left for the reader to fill in. There are a lot of exercises included, ranging from the straightforward to more challenging ones. Some of them include a bit of implementation and some experiments to be done on the computer. We strongly encourage students not to skip these parts. In addition there are some "projects." These are either included to refresh the student's memory of results needed in this course, or to extend the theories developed in the present text.

Given the fact that we introduce both numerical and analytical tools, we have chosen to put little emphasis on modeling. Certainly, the derivation of models based on partial differential equations is an important topic, but it is also very large and can therefore not be covered in detail here.

The first seven chapters of this book contain an elementary course in partial differential equations. Topics like separation of variables, energy arguments, maximum principles, and finite difference methods are discussed for the three basic linear partial differential equations, i.e. the heat equation, the wave equation, and Poisson's equation. In Chapters 8–10 more theoretical questions related to separation of variables and convergence of Fourier series are discussed. The purpose of Chapter 11 is to introduce nonlinear partial differential equations. In particular, we want to illustrate how easily finite difference methods adopt to such problems, even if these equations may be hard to handle by an analytical approach. In Chapter 12 we give a brief introduction to the Fourier transform and its application to partial differential equations.

Some of the exercises in this text are small computer projects involving a bit of programming. This programming could be done in any language. In order to get started with these projects, you may find it useful to pick up some examples from our web site, http://www.ifi.uio.no/~pde/, where you will find some Matlab code and some simple Java applets.

Acknowledgments

It is a great pleasure for us to thank our friends and colleagues for a lot of help and for numerous discussions throughout this project. In particular, we would like to thank Bent Birkeland and Tom Lyche, who both participated in the development of the basic ideas underpinning this book. Also we would like to thank Are Magnus Bruaset, Helge Holden, Kenneth Hvistendahl Karlsen, Jan Olav Langseth, Hans Petter Langtangen, Glenn Terje Lines, Knut Mørken, Bjørn Fredrik Nielsen, Gunnar Olsen, Klas Samuelsson, Achim Schroll, Wen Shen, Jan Søreng, and Åsmund Ødegård for reading parts of the manuscript. Finally, we would like to thank Hans Birkeland, Truls Flatberg, Roger Hansen, Thomas Skjønhaug, and Fredrik Tyvand for doing an excellent job in typesetting most of this book.

Oslo, Norway, July 1998

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Contents

1
Setting the Scene

You are embarking on a journey in a jungle called Partial Differential Equations. Like any other jungle, it is a wonderful place with interesting sights all around, but there are also certain dangerous spots. On your journey, you will need some guidelines and tools, which we will start developing in this introductory chapter.

1.1 What Is a Differential Equation?

The field of differential equations is very rich and contains a large variety of different species. However, there is one basic feature common to all problems defined by a differential equation: the equation relates a function to its derivatives in such a way that the function itself can be determined. This is actually quite different from an algebraic equation, say

$$x^2 - 2x + 1 = 0,$$

whose solution is usually a number. On the other hand, a prototypical differential equation is given by

$$u'(t) = u(t).$$

The solution of this equation is given by the function

$$u(t) = ce^t,$$

where the constant c typically is determined by an extra condition. For instance, if we require

$$u(0) = 1/2,$$

we get $c = 1/2$ and $u(t) = \frac{1}{2}e^t$. So keep this in mind; the solution we seek from a differential equation is a function.

1.1.1 Concepts

We usually subdivide differential equations into partial differential equations (PDEs) and ordinary differential equations (ODEs). PDEs involve partial derivatives, whereas ODEs only involve derivatives with respect to one variable. Typical ordinary differential equations are given by

$$\begin{aligned}
(a)\ u'(t) &= u(t),\\
(b)\ u'(t) &= u^2(t),\\
(c)\ u'(t) &= u(t) + \sin(t)\cos(t),\\
(d)\ u''(x) + u'(x) &= x^2,\\
(e)\ u''''(x) &= \sin(x).
\end{aligned} \tag{1.1}$$

Here (a), (b) and (c) are "first order" equations, (d) is second order, and (e) is fourth order. So the *order* of an equation refers to the highest order derivative involved in the equation. Typical partial differential equations are given by[1]

$$\begin{aligned}
(f)\ u_t(x,t) &= u_{xx}(x,t),\\
(g)\ u_{tt}(x,t) &= u_{xx}(x,t),\\
(h)\ u_{xx}(x,y) + u_{yy}(x,y) &= 0,\\
(i)\ u_t(x,t) &= \big(k(u(x,t))u_x(x,t)\big)_x,\\
(j)\ u_{tt}(x,t) &= u_{xx}(x,t) - u^3(x,t),\\
(k)\ u_t(x,t) + \big(\frac{1}{2}u^2(x,t)\big)_x &= u_{xx}(x,t),\\
(l)\ u_t(x,t) + (x^2+t^2)u_x(x,t) &= 0,\\
(m)\ u_{tt}(x,t) + u_{xxxx}(x,t) &= 0.
\end{aligned} \tag{1.2}$$

Again, equations are labeled with orders; (l) is first order, (f), (g), (h), (i), (j) and (k) are second order, and (m) is fourth order.

Equations may have "variable coefficients," i.e. functions not depending on the unknown u but on the independent variables; t, x, or y above. An equation with variable coefficients is given in (l) above.

[1] Here $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and so on.