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Optimal Control of Constrained Piecewise Affine Systems

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... for M. & J.

Noise is the key; life is full of noise. Only death is silent. ALEC EMPIRE

PREFACE

One of the most important and challenging problems in control is the derivation of systematic tools for the computation of controllers for general constrained nonlinear or *hybrid systems* [vS00] (combining continuous-valued dynamics with logic rules) that can guarantee (among others) closed-loop stability, feasibility, and optimality with respect to some objective function.

The focus of this book lies on the class of constrained discrete-time *piecewise affine* (PWA) *systems* [Son81]. This particular subclass of hybrid systems is obtained by partitioning the extended state-input space into regions and associating a different affine state update equation with each region. Under mild assumptions this modeling framework is equivalent to many other hybrid system classes reported in the literature [HDB01], and represents itself as a powerful modeling tool to capture general nonlinearities (e.g. by local approximation), constraints, saturations, switches, discrete inputs, and other hybrid modeling phenomena in dynamical systems.

Although piecewise affine systems are a subclass of general nonlinear systems, most of the (classical) control theory developed for nonlinear control does not apply, due to commonly made continuity and smoothness requirements.

The flexibility of this modeling framework and the recent technological advances in the fields of optimization and control theory have lead to a considerable interest in academia and industry in piecewise affine systems; not to mention that many engineering systems naturally express themselves or approximate nicely as piecewise affine systems, or can be 'straightforwardly' translated into them.

Simple examples are mechanical systems with backlash, where the system has a different dynamical behavior in the 'contact mode' than in the 'non-contact mode', or mechanical systems with dry friction (e.g. car tires), where the system is in the stick or slip mode, depending on the force acting on the systems. Another example is a chemical process that abruptly changes its phase, for instance, from the gaseous to the liquid phase, depending on the temperature or pressure prevailing in the system.

Naturally, the computation of stabilizing feedback controllers for constrained piecewise affine systems remains to be a challenging task, especially for fast sampled systems (i.e. in the range of milli- or microseconds).

The work at hand revolves around the efficient and systematic computation, analysis, and post-processing of *closed-form*, stabilizing, optimal, exact state-feedback controllers for these systems, where the cost function of the respective optimal control problem is composed of (piecewise) linear vector norms. For the here considered control problems, the underlying *constrained finite time optimal control* (CFTOC) problem can be solved by means of *multi-parametric (mixed-integer) linear programming* [Bor03, DP00]. Once the closed-form optimal solution is computed, the resulting controller can be utilized as optimal control lookup table of the current measured state. Thus the on-line computational effort reduces to a simple evaluation of the measured state in the lookup table to obtain the optimal control action.

In addition, this lookup table can be implemented for example into a microprocessor and cheaply be replicated in mass production. This is in contrast to the alternative on-line optimization counterpart, which usually implies expensive and large computational infrastructure or a limitation of the complexity of the corresponding optimization problem to cope with the possibly high sampling rates.

Outline

This book is structured as follows: Chapters 1 and 2 present some necessary background material on the mathematical terminology, the constrained (parametric) optimization, and systems and control theory, used throughout this manuscript.

The most successful modern control strategy both in theory and in practice for constrained systems is undoubtedly *Receding Horizon Control* (RHC) [MRRS00, Mac02], often also interchangeably called *Model (Based) Predictive Control* (MPC). In Chapter 3 the basic idea of receding horizon control is introduced, the underlying constrained finite time optimal control problem is explained and the corresponding closed-form solution and its usage is indicated.

Chapter 4 defines the discrete-time piecewise affine (PWA) system class that is the main focus of the whole work at hand.

The *constrained finite time optimal control* (CFTOC) problem with a (piecewise) linear vector norm based cost function for the class of PWA systems, and its use in the context of receding horizon control, is described in Chapter 6. One way to obtain the closed-form solution to this problem is by reformulating the problem into its equivalent mixed logical dynamical form and solving a multi-parametric mixed-integer linear program (mp-MILP) [Bor03, DP00]. This, however, is in general a computationally challenging task. Here a novel algorithm, which combines a dynamic programming strategy with a multi-parametric linear program solver and some basic polyhedral manipulation, is presented and compared to the aforementioned mp-MILP approach. By comparison with results in the literature it is shown that the presented algorithm solves the considered class of problems in a computationally more efficient manner.

Chapter 7 extends the ideas presented in Chapter 6 to the case of *constrained infinite time optimal control* (CITOC) problems for the same general system class. The equivalence of the dynamic programming generated solution with the solution to the infinite time optimal control problem is shown. Furthermore, new convergence results of the dynamic programming strategy for general nonlinear systems and stability guarantees for the resulting possibly *discontinuous* closed-loop system are given. A computationally efficient algorithm is presented, which obtains the closed-form infinite time optimal solution, based on a particular dynamic programming exploration strategy with a multi-parametric linear programming solver and basic polyhedral manipulation. Intermediate solutions of the dynamic programming strategy give stabilizing suboptimal controllers with guaranteed optimality bounds.

In Part III of this manuscript the further focus lies on the analysis and post-processing techniques for the closed-form optimal controllers obtained in Part II, i.e. Chapter 6 and 7.

Chapter 8 continues on the results proposed in [Pol95, KAS92] for computing linear vector norm based Lyapunov functions for linear discrete-time systems. The finitely terminating algorithm builds on a decomposition procedure and utilizes the solution of a finite and bounded sequence of feasibility LPs with very few constraints or, equivalently, very simple algebraic tests. The here computed weight W of the weighted vector norm is of small size and all the considered steps during the construction are elementary and scale well for large systems. Such Lyapunov functions are commonly utilized for a priori ensuring closed-loop asymptotic or exponential stability and all time feasibility when using a receding horizon control policy as considered in Chapter 3 and 6. For this purpose a simple algorithm for a class of PWA systems is presented.

In general, without extending or modifying the underlying optimization problem, stability and/or feasibility for all time of the closed-loop system is not guaranteed a priori, when a receding horizon control strategy is used. In Chapter 9 an algorithm is presented that by analyzing the CFTOC solution of a PWA system a posteriori extracts regions of the state space for which closed-loop stability and feasibility for all time can be guaranteed. The algorithm computes the maximal positively invariant set and stability region of a piecewise affine system by combining reachability analysis with some basic polyhedral manipulation. The simplicity of the overall computation stems from the fact that in all steps of the algorithm only linear programs need to be solved.

In Chapter 10 the concept of the stability tube is presented. By a posteriori analyzing a nominal stabilizing controller and the corresponding Lyapunov function of a general closed-loop nonlinear system one can compute a set, called stability tube, in the augmented state-input space. Any control chosen from this set guarantees closed-loop stability and a pre-specified level of closed-loop performance. Furthermore, these stability tubes can serve, for example, as a robustness analysis tool for the given nominal controller as well as a means to obtain an approximation of the nominal control law with lower complexity while preserving closed-loop stability and a certain performance level. For the PWA systems and PWA functions, considered in this manuscript, the overall computation of the stability tubes reduces to basic polytope manipulations.

The on-line evaluation of the above mentioned closed-form state feedback control law, i.e. the optimal lookup table, requires the determination of the state space region in which the measured state lies, in order to decide which 'piece' of the control law to apply. This procedure is called the *point location problem*, and the rate at which it can be solved determines the maximum sampling speed of the system allowed. In Chapter 11, a novel and computationally efficient algorithm based on bounding boxes and an n-dimensional interval tree is presented that significantly improves this point-location search for piecewise state feedback control laws defined over a large number of (possibly overlapping) polyhedra, where the required off-line preprocessing is low and so the approach can be applied to very large systems, which is substantiated by numerical examples.

Note on This Manuscript

This book is a revised and extended version of the author's PhD thesis [Chr06] written at the Automatic Control Laboratory of ETH Zurich in Switzerland. Parts of this manuscript are based on previously published articles. Note that most of the results in this manuscript were obtained in close collaboration with various colleagues. The appropriate cited references and list of authors throughout the text reflect this fact.

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Frank J. Christophersen
Zurich and Zagreb, 2006/'07

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NOTATION

Throughout this book, as a general rule, scalars and vectors are denoted with lower case letters ($a, b, \ldots, \alpha, \beta, \ldots$), matrices are denoted with upper case letters ($A, B, \ldots$), and sets are denoted with upper case calligraphic letters ($\mathcal{A}, \mathcal{B}, \ldots$).

In the following let $i, j, m, n \in \mathbb{N}$ be integers such that $i \leq m, j \leq n$, let $s \in \mathbb{R}^n$ be a column vector, $S \in \mathbb{R}^{n \times n}$ be a square matrix, $R \in \mathbb{R}^{m \times n}$ be a rectangular matrix, $\mathcal{I} \subseteq \{1, \ldots, m\}$ be a set of integers, and $\mathcal{S} \subseteq \mathbb{R}^n$ be a subset or collection of subsets of an n-dimensional real vector space.

General Operators, Relations, and Functions

$\cdot$	general placeholder (for any variable, index, set, or space)
$\tilde{\cdot}$	function, variable, set, etc. that was derived or is in a special relationship to $\cdot$
$\breve{\cdot}$	generic object (set, function, etc.)
$\ldots$	"and so forth"
$:=$	left-hand side is defined by the right-hand side
$=:$	right-hand side is defined by the left-hand side
$\mid$	such that
$:$	such that

$\in$	is element of (belongs to)
$\forall$	for all
$\exists$	there exists at least one
$\exists!$	there exists exactly one
$\notin, \nexists, \ldots$	/ denotes negation
$\cdot \rightarrow \cdot$	mapping, "maps from ... to ..."
$\{\cdot, \cdot, \ldots\}$	a set or sequence
$(\cdot, \cdot, \ldots)$	a composite variable (ordered list, tuple), e.g. $(s, i, \mathcal{S}) \in \mathbb{R}^n \times \mathbb{N} \times 2^{\mathbb{R}^m}$
$\vee$	or
$\wedge$	and
$\Rightarrow$	implies
$\Leftarrow$	is implied by
$\Leftrightarrow$	equivalence, "if and only if"
$\mathring{f}(\cdot)$	restriction of the function $f(\cdot)$ to the neighborhood around the origin
$\deg(\cdot)$	degree of a polynomial
$\mathrm{dom}(\cdot)$	domain of a function/mapping
$\mathrm{range}(\cdot)$	range of a function/mapping
$\lceil \cdot \rceil$	ceiling function, i.e. smallest integer greater or equal than its argument
$\lfloor \cdot \rfloor$	floor function, i.e. greatest integer less or equal than its argument
j	$\sqrt{-1}$
e	Euler number, $\mathrm{e} \approx 2.71828182845905$

Sets and Spaces

$\{\cdot, \ldots\}$	set or sequence
$[\cdot, \cdot)$	interval on the real line, where here the lower endpoint is included (closed) and the upper endpoint in not (open)
$\mathcal{B}_\varepsilon(x)$	n_x-dimensional ε-ball around $x \in \mathbb{R}^{n_x}$
$\mathcal{N}(\mathcal{S})$	neighborhood of the set $\mathcal{S}$
$\emptyset$	empty-set
$\mathbb{C}$	complex numbers
$\mathbb{C}^n$	space of n-dimensional (column) vectors with complex entries
$\mathbb{C}^{m \times n}$	space of m by n matrices with complex entries

$\mathbb{R}$	real numbers
$\mathbb{R}^n$	space of n-dimensional (column) vectors with real entries
$\mathbb{R}^{m\times n}$	space of m by n matrices with real entries
$\mathbb{R}_{\geq c}$	real numbers $\geq c$, i.e. $\mathbb{R}_{\geq c} := \{x \in \mathbb{R} \mid x \geq c\}$
$\mathbb{Q}$	rational numbers
$\mathbb{Z}$	integers
$\mathbb{Z}_{\geq c}$	integers $\geq c$, i.e $\mathbb{Z}_{\geq c} := \{x \in \mathbb{Z} \mid x \geq c\}$
$\mathbb{N}$	natural numbers (positive integers), $\mathbb{N} = \mathbb{Z}_{>0}$
$\mathbb{N}_{\geq c}$	natural numbers $\geq c$, i.e $\mathbb{N}_{\geq c} := \{x \in \mathbb{N} \mid x \geq c\}$

Set Operators

$(\subset)\ \subseteq$	(strict) subset
$(\supset)\ \supseteq$	(strict) superset
$\cap$	intersection
$\cup$	union
$\setminus$	set difference
$\times$	Cartesian product, $\mathcal{X} \times \mathcal{Y} = \{(x,y) \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$
$\lvert\mathcal{I}\rvert$	number of elements (cardinality) of a set $\mathcal{I}$
$N_{\mathcal{S}}$	number of polytopes (or sets) in the collection of polytopes (resp. sets), $\mathcal{S} = \{\mathcal{S}_i\}_{i=1}^{N_{\mathcal{S}}}$
$2^{\mathcal{S}}$	power set (set of all subsets of $\mathcal{S}$)
$\mathcal{S}_i$	i-th polytope (or set) of a collection of polytopes (resp. sets) $\mathcal{S} = \{\mathcal{S}_i\}_{i=1}^{N_{\mathcal{S}}}$
$\mathring{\mathcal{S}}$	collection of sets $\mathcal{S}_i$ for which $\mathbb{0} \in \mathcal{S}_i$ (neighborhood set around the origin)
$\partial\mathcal{S}$	boundary of $\mathcal{S}$
$\bar{\mathcal{S}}$	closure of $\mathcal{S}$
$\underline{\mathcal{S}}$	underlying set of $\mathcal{S}$
$\mathrm{conv}(\cdot)$	convex hull
$\mathrm{aff}(\cdot)$	affine hull
$\dim(\cdot)$	dimension
$\mathrm{affdim}(\cdot)$	affine dimension
$\mathrm{int}(\cdot)$	strict interior
$\mathrm{relint}(\cdot)$	relative interior
$\mathrm{vert}(\cdot)$	set of vertices
$\mathrm{vol}(\cdot)$	volume
$\mathrm{face}_i(\mathcal{S})$	i-th $(n-1)$-dimens. face of the n-dimensional polytope $\mathcal{S}$
$\mathrm{face}_i^d(\mathcal{S})$	i-th d-dimensional face of the n-dimensional polytope $\mathcal{S}$

$\text{proj}_{\mathcal{P}}(\cdot)$	projection onto the constraint or set $\mathcal{P}$
$\text{proj}_x(\cdot)$	projection onto the x-subspace

Operators on Vectors and Matrices

$\mathbb{0}_n$	vector of zeros, $\mathbb{0}_n := [0\,0\,\ldots\,0]' \in \mathbb{R}^n$
$\mathbb{1}_n$	vector of ones, $\mathbb{1}_n := [1\,1\,\ldots\,1]' \in \mathbb{R}^n$
e_i, e_i^n	i-th column of the unit-matrix I (resp. I_n)
$[\cdot,\cdot,\ldots]$	a matrix (or a vector)
$<, \leq, =, \geq, >$	element-wise comparison of vectors
$\text{diag}(s)$	diagonal matrix with diagonal elements $[\text{diag}(s)]_{i,i} = s_i$
s'	row vector, transpose of a vector
s_i, $[s]_i$	i-th element of a vector (second notation only if ambiguous)
$s_{\mathcal{I}}$, $[s]_{\mathcal{I}}$	vector formed from the elements indexed by $\mathcal{I}$
$\|s\|$	element-wise absolute value
$\|\|s\|\|$	(any) vector norm
$\|\|s\|\|_p$	vector p-norm (Hölder norm)
$\|\|s\|\|_1$	vector 1-norm (sum of absolute elements of a vector, Manhattan norm)
$\|\|s\|\|_2$	vector 2-norm (Euclidian norm)
$\|\|s\|\|_\infty$	vector ∞-norm (largest absolute element of a vector)

0	zero matrix (of appropriate dimension)
$0_{m\times n}$	zero matrix of dimension $m \times n$
I	identity matrix (of appropriate dimension)
I_n	identity matrix of dimension $n \times n$
$[\cdot,\cdot,\ldots]$	a matrix (or a vector)
$R_{i,j}$, $[R]_{i,j}$	(i,j)-th element of a matrix (second notation only if ambiguous)
$[R]_i$, $[R]_{i,\cdot}$	i-th row of a matrix
$[R]_{\mathcal{I}}$	matrix formed from the rows indexed by $\mathcal{I}$
$[R]_{\cdot,j}$	j-th column of a matrix
$[R]_{\cdot,\mathcal{J}}$	matrix formed from the columns indexed by $\mathcal{J}$
$S\ (\succeq) \succ 0$	positive (semi)definite matrix
$S\ (\preceq) \prec 0$	negative (semi)definite matrix
R'	transpose of a matrix
R^{H}	conjugate transpose (Hermitian adjoint) of a matrix
S^{-1}	inverse of the square matrix
$\det(S)$	determinant of the square matrix
$\text{diag}(S)$	vector composed of the diagonal elements of a matrix, $[\text{diag}(S)]_i = S_{i,i}$