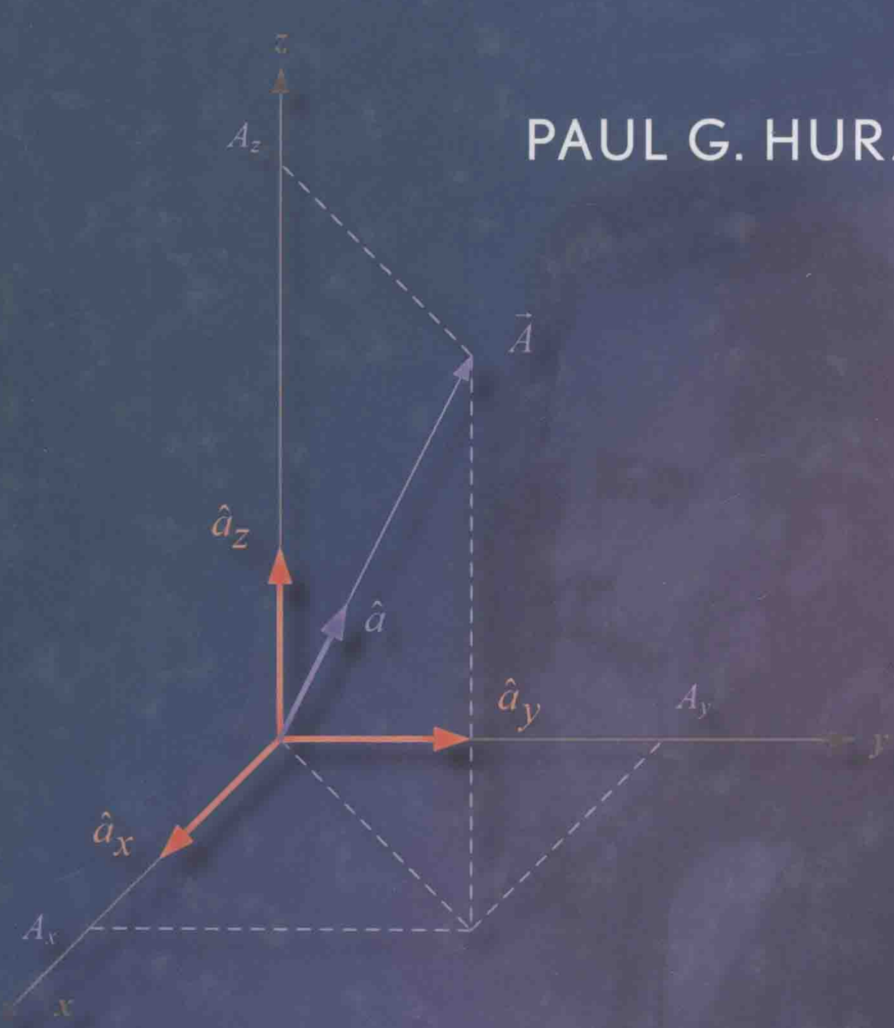
MAXWELL'S EQUATIONS



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Maxwell's Equations

Paul G. Huray

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A John Wiley & Sons, Inc., Publication

Published by John Wiley & Sons, Inc., Hoboken, New Jersey
Published simultaneously in Canada

Library of Congress Cataloging-in-Publication Data:

Huray, Paul G., 1941–
Maxwell's equations / Paul G. Huray.
p. cm.
Includes bibliographical references.
ISBN 978-0-470-54276-7 (cloth)
1. Maxwell equations. I. Title.
QC670.H87 2010
530.14'1–dc22

2009019340

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

Maxwell's Equations

This book is dedicated to the author's lifelong partner
Susan Lyons Huray

Acknowledgments

The author owes a debt of gratitude to Dr. Yinchao Chen of the Electrical Engineering department at the University of South Carolina (USC). Dr. Chen has published articles with the author and has had many discussions on the techniques and meaning of the solutions to Maxwell's equations and their applications. Other USC professors who contributed to the physical and chemical understanding of printed circuit board (PCB) materials were Michael Myrick of the Chemistry Department and Richard Webb of the Physics and Astronomy Department.

The author, Dr. Chen, and three Signal Integrity engineers (Brian Knotts, Hao Li, and Richard Mellitz) from the Intel Corporation in Columbia, SC, created the first graduate Signal Integrity program in 2003, which has since produced more than 80 practicing Signal Integrity engineers, many of whom read and corrected early drafts of this text.

The author conducts industrial research part time with the Intel Corporation in the area of high-speed electromagnetic signals. In this work, he has had the privilege to work closely with Richard Mellitz and Stephen Hall, on applications of electromagnetism for practical use. It was their penetrating questions that prompted many of the explanations in this text. Another Intel employee, Dan Hua, provided a sequence of exchanged papers on the evaluation of scattering and absorption in the language of vector spherical harmonics; it was through these discussions that the sections on absorption by small good conducting spheres arose. Gary Brist taught the author (and many of his graduate students) about the process of manufacturing PCB stack-ups and stimulated many of the questions that are sprinkled throughout the book. Anusha Moonshiram and Chaitanya Sreerema conducted many of the high-frequency vector network analyzer (VNA) measurements in this text. Femi Oluwafemi conducted many of the numerical simulations on phase analysis to identify time-dependent fields inside good conductors and provided many of the final comparisons to the VNA data. Guy Barnes and Paul Hamilton provided the Fabry–Perot measurements of permittivity. Brandon Gore helped work on magnetic losses, and David Aerne assisted the analysis of spherical composition profiles and near neighbor interference effects. Peng Ye was a sounding board for arguments about the analytical analysis associated with electromagnetic field dynamics. Kevin Slattery introduced the author to near-field scanning electromagnetic probes and helped direct the work of two USC graduate students, Jason Ramage and Christy Madden Jones, whose work on proof of Snell's law at microwave frequencies and absorption by impurities appears in the text. Intel engineers such as Howard Heck, Richard Kunze, Ted Ballou, Steve Krooswyk, Matt Hendrick, David Blakenbeckler, and Johnny Gibson passed through USC during the writing of this text to present lectures to the author's Signal Integrity classes and to build richness into the intellectual atmosphere. Mark Fitzmaurice was always ready to help make the Signal Integrity program at USC a success through his support for measurement equipment, student internships, and common sense.

Many USC undergraduate and graduate students contributed to the testing and writing of this book. Steven Pytel worked with the author on scanning electron microscope (SEM) and analysis measurements at the Oak Ridge National Laboratory in Oak Ridge, TN, and, while working for Intel, was the sounding board for many of the arguments presented here. After receiving his PhD, he became an employee of the Ansoft Corporation of Pittsburg, PA, where he became an applications engineer for Signal Integrity tools. He is primarily responsible for the material in Chapter 8 of the author's book *The Foundations of Signal Integrity* on numerical simulations. Ken Young helped with editing, Fisayo Adepetun provided assistance with figures and the reduced magnetic moment analysis, and David London supported web pages for testing and transmittal of the chapters. Tom McDonough gave lectures to the Signal Integrity classes on the use of Synopsys Corporation of Mountain View, CA, HSpice software and helped in the analysis of ceramic capacitor fields.

John Fatcheric of the Oak Mitsui Corporation of Camden, SC, assisted the presentation on copper surface production. Bob Helsby, Charles Banyon, Arien Sligar, and Zol Cendes of the Ansoft Corporation of Pittsburg, PA, supported the use of forefront numerical solutions to Maxwell's equations. James Rautio of Sonnet Software in Syracuse, NY, assisted on the history Maxwell and the use of his portrait. Mike Resso of Agilent Corporation of Santa Clara, CA, supported a joint Intel–Agilent VNA donation. Lee Riedinger, Harry M. Meyer III, Larry Walker, and Marc Garland of the Oak Ridge National Laboratory assisted in making qualitative and quantitative measurements of PCB components by SEM and Auger analysis. José E. Rayas Sánchez of the Instituto Technológico y de Estudios Superiores de Occidente (ITESO) at the Jesuit University of Guadalajara, Mexico, James Gover of Kettering University in Flint, MI, and John David Jackson of University of California-Berkeley and Lawrence Berkeley National Laboratory, in Berkeley, CA, provided discussions on Maxwell's interpretations and Signal Integrity of high-speed circuits.

Introduction

James Clerk Maxwell

Maxwell's Equations[i]

Differential form	Integral form
$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$
$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_C \vec{H}\cdot d\vec{l} = I + \oint_S \frac{\partial \vec{D}}{\partial t}\cdot d\vec{s}$
$\vec{\nabla}\cdot\vec{D} = \rho_V$	$\oint_C \vec{D}\cdot d\vec{s} = Q$
$\vec{\nabla}\cdot\vec{B} = 0$	$\oint_S \vec{B}\cdot d\vec{s} = 0$

BIOGRAPHY

James Clerk Maxwell was born on June 13, 1831, in Edinburgh, Scotland, and was educated at his country home until he was 8 years old, when his mother died. At age 10, he attended Edinburgh Academy, where he was given the nickname[ii] "Dafty." At age 16, he attended the University of Edinburgh, and, at age 19, he went to Peterhouse Cambridge but moved to Trinity to obtain a fellowship. In 1856, he moved to Marischal College in Aberdeen to be near his father, who shortly thereafter died. Maxwell married Katherine Mary Dewar, the daughter of the Principal of Marischal College, in 1859. In 1860, Maxwell was appointed to the chair of Natural Philosophy at King's College in London, where he did his most productive work, and, from 1871 until his death on November 5, 1879, he was the first Cavendish Professor at Cambridge, where he produced his two-volume treatise,[iii] created a working laboratory, and edited Henry Cavendish's researches for publication. Albert

Einstein once described Maxwell's work[iv] as the "most profound and the most fruitful that physics has experienced since the time of Newton."

ELECTROMAGNETIC CONCEPTS IN THE MID-1800s

John Tyndall wrote a series of essays, addresses, and reviews entitled *Fragments of Science*,[v] which began with the copy of a 1865 essay on "The Constitution of Nature,"[vi] in which the perspective of "modern" science described electromagnetics as follows:

> *From the phenomena of sound, as displayed in the air, "men's minds" ascend to the phenomena of light, as displayed in the ether; which is the name given to the interstellar medium. The notion of this medium must not be considered as vague or fanciful conception on the part of scientific men. Of its reality most of them are as convinced as they are of the existence of the sun and the moon. The luminiferous ether has definite mechanical properties. It is almost infinitely more attenuated than any know gas, but its properties are those of a solid rather than of a gas. It resembles jelly rather than air.*

It was the revolutionary thinking of Einstein that changed that description to a relativistic philosophy that there is no luminiferous ether, but, rather, the speed of light is the same for all observers, independent of their own velocity relative to the source. Einstein was one of the first scientists to adopt the rubric of the observer for logically deducing outcomes, but it was an extension of one of the fundamental principles of science[vii] that this text has used to develop a process of explaining physical processes: "Upon every particle there exists an observer and that observer is you." Students in this endeavor are asked to "think like an electromagnetic wave" for the purpose of finding a logical solution.

In an article on "Scientific Materialism"[viii] in 1868, Tyndall also noted

> *Mathematics and physics have been long accustomed to coalesce, and here they form a single section. No matter how subtle a natural phenomenon may be, whether we observe it in the region of sense, or follow it into that of imagination, it is in the long run reducible to mechanical laws. But the mechanical data once guessed or given, mathematics are all-powerful as an instrument of deduction. The command of Geometry over the relations of space, and the far-reaching power which Analysis confers, are potent both as a means of physical discovery, and of reaping the entire fruits of discovery.*

In this book, we follow one of the most beautiful and fruitful applications of mathematics that has ever been written. It is the culmination of many individual mathematicians and scientists that has led us to the solutions of Maxwell's equations. Most of these early mathematicians were not interested in applications, and most did their work prior to the development of Maxwell's equations. French, German, and English academicians such as Bessel, Cauchy, Clebsch, Dirichlet, Fourier, Gauss, Green, Hankel, Helmholtz, Hermite, Hilbert, Laguerre, Laplace, Laurent, Legendre, Lorentz, Lorenz, Neumann, Poisson, Rayleigh, Schmidt, Stokes and Wronski built a foundation of solutions for partial differential equations that stood alone. When Sturm and Liouville categorized their work into a common framework, they were able to demonstrate that a set of complete, orthogonal, functions could uniquely describe a set of eigenfunctions that guaranteed a solution to Maxwell's equations. This framework is one of the most elegant sets of solutions ever to be devised, and we are the beneficiaries of their work. We shall adopt their efforts with reference to their work. Relevant texts for this systematic approach are given by Arfken,[ix] and Jackson,[x] and a self-study set of outlines is available from Spiegel.[xi]

Rather than focus on mathematical proofs of solutions, this book is intended to be a foundation for the discipline of electricity and magnetism upon which measurements, simulations, and "rules-of-thumb" are built. In that sense, it is intended to be a book that takes theory to practice. It is written in the language of an electrical engineer rather than a mathematician or physicist and is intended to support engineering practice.[xii]

Practical work has provided a dilemma for scientists since the time of Maxwell. For example, following an essay "On Prayer as a Form of Physical Energy," Tyndall includes a comment on the popular interest in understanding electromagnetics:

> *It is the custom of the Professors in the Royal School of Mines in London to give courses of evening lectures every year to working men. The lecture-room holds 600 people; and tickets to this amount are disposed of as quickly as they can be handed to those who apply for them. So desirous are the working men of London to attend these lectures that the persons who fail to obtain tickets always bear a large proportion to those who succeed. Indeed, if the lecture-room could hold 2,000 instead of 600, I do not doubt that every one of its benches would be occupied on these occasions. The information acquired is hardly ever of a nature which admits of being turned into money. It is therefore, a pure desire for knowledge, as a good thing in itself, and without regard to its practical application, which animates the hearers of these lectures.*

Tyndall concludes his Fragments of Science with the following insight:

> *Two orders of minds have been implicated in the development of this subject; first, the investigator and discoverer, whose object is purely scientific, and who cares little for practical ends; secondly, the practical mechanician, whose object is mainly industrial. It would be easy, and probably in many cases true, to say that the one wants to gain knowledge, while the other wishes to make money; but I am persuaded that the mechanician not infrequently merges the hope of profit in the love of his work.*

EXTENSION OF ELECTROMAGNETIC THEORY INTO THE 2000s

One of the great ironies of history is that Maxwell's equations were not written by Maxwell, at least not in their vector form stated on the first page. Maxwell was convinced that the laws of electromagnetism would be best formulated in the form of a quaternion, which had been invented by the Irish mathematician Sir William Rowan Hamilton in 1843 because they worked in four dimensions and could therefore include three-dimensional space and time. The original form of Maxwell's equations was thus in the form of 20 quaternion expressions that will be discussed in the vector chapter, which included eight equations dealing with electromagnetic fields that include the magnetic vector potential and 12 that deal with the magnetic scalar potential, magnetic mass, and magnetic conductivity. Some scientists today[xiii] say that the vector-reformulated equations by Heavyside, Gibbs, Fitzgerald, Lodge, and Hertz are insufficient to describe some physical outcomes such as the Aharonov–Bohm effect, the Josephson effect, the quantum Hall effect, the De Hass Van Alphen effect, and the Sagnac effect, all of which make the magnetic scalar and vector potentials (sometimes called the A_μ fields with $\mu = 0, 1, 2, 3$) physically meaningful constructs. The Heavyside formulation (and this book) treats the magnetic vector field as a mathematical convenience. As Lord Kelvin said in 1892, "Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Clerk Mawell." Maxwell's first formulation of a magnetic charge density and the possible existence of magnetic monopoles were forgotten for half a century until P. A. M. Dirac again speculated on their existence in 1931. In deference to my former colleague, Yakir Aharonov, these potentials are mentioned extensively in this book but usually in the vein of an engineer's interpretation that they give the correct mathematical result but show few practical macroscopic properties and do not interact directly with particles. A few scientists like H. Harmuth and K. Meyl go a step further to state that there exist no kind of monopoles, electric or magnetic; that the alleged electric monopoles (charges) are only secondary effects of electric and magnetic fields. We will see that this description of fields as wavelike entities is contained in the dual nature of matter as either particle or wavelike, depending on the interpretation for a given application. Both descriptions have a disconcerting problem: the magnetic vector potentials require "action at a distance" and that electromagnetic history influences current properties, whereas the vector field interpretation requires that waves propagate in luminiferous ether that has no physically measurable properties.

INTENT OF THE BOOK

It is the intent of this book to first clarify the concept of Tyndall's jelly, which transmits "action at a distance" (as it was called in 1865). This portion of the book is a mathematical and physical treatment that enables "working men" (and women) to understand the greatest set of equations ever devised,[1] now called Maxwell's equations. It focuses on our "pure desire for knowledge" and is intended to permit the readers to convince themselves that Maxwell's equations provide a framework for a multitude of application. In the current book, techniques that show how to obtain analytic solutions to Maxwell's equations for ideal materials and boundary

[1] In 2004, Physics Web readers voted Maxwell's equations and Euler's equation to be the greatest equations of all time.

conditions are presented. These solutions are then used as a benchmark for the student to solve "real world" problems via computational techniques, first confirming that a computational technique gives the same answer as the analytic solution for an ideal problem. A subsequent book, *The Foundations of Signal Integrity*,[xiv] concentrates on the solutions to Maxwell's equations in a variety of media (various flavors of jelly) and with a variety of boundary conditions.

This information is presented to the twenty-first-century students[2] in the hope that they will consider mathematical and physical concepts as *integral*. The students are challenged to not accept uncertainty but to be honest with themselves in appreciating and understanding the derivations of the electromagnetic giants. After the mathematical solution has been obtained, we hope that the students will ask, "What are these equations telling me?" and "How could I use these in some other application?" Perhaps the students will delve even deeper to ask, "What are the physical phenomena that cause fields to exist, to move, to reflect, or to transmit through materials?" With such an armada of knowledge, the students can take these electromagnetic concepts to further applications and to further "stand on the shoulders of giants"[3] (perhaps for monetary gain).

And while the students may criticize the concept of luminiferous ether and make fun of the ancient practice of including an essay on *Prayer* in the context of natural law, they should review the beauty of such a set of symmetric equations named for Maxwell. It is worthy of note that, while we call these equations Maxwell's equations, a student might ask, "*Why* do these equations describe nature in such a simple form?" "Is it possible that the mind of man is incapable of understanding or postulating a more complex set of equations?" or "Is it not possible that nature is so mathematically beautiful because God made it that way for our pleasure?"

NOTES

i. James Clerk Maxwell, "A Dynamical Theory of the Electromagnetic Field," *Philosophical Transactions of the Royal Society of London* 155 (1865): 459–512. Oliver Heaviside reformulated Maxwell's equations (originally in quaternion format) to this asymmetric vector form. In Chapter 7, concepts of electric vector potential **and** magnetic vector potential are shown to make the equations fully symmetric.

ii. http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Maxwell.html

iii. James Clerk Maxwell, *A Treatise on Electricity & Magnetism*, Vol. 1, unabridged 3rd ed. (New York: Dover, 1954), Vol. 2 (La Verne, CA: Merchant Books, 2007).

[2] One reader from the Physics Web polled that rated Maxwell's equations as the most beautiful equations ever derived recalled how he learned Maxwell's equations during his second year as an undergraduate student, "I still vividly remember the day I was introduced to Maxwell's equations in vector notation," he wrote. "That these four equations should describe so much was extraordinary…For the first time I understood what people meant when they talked about elegance and beauty in mathematics or physics. It was spine-tingling and a turning point in my undergraduate career."

[3] The quote "If I have seen farther than others, it is because I have stood on the shoulders of giants" was attributed to Sir Issac Newton because it appeared in a letter he wrote to Robert Hooke in 1675 but was also used by an eleventh-century monk named John of Salisbury, and there is evidence he may have gotten it from an older text while studying with Abelard in France.

iv. Paul Arthur Schilpp, ed., *Albert Einstein: Philosopher-Scientist* (La Salle, IL: Open Court, 1951), 63. Einstein once wrote, "The special theory of relativity owes its origin to Maxwell's equations of the electromagnetic field."

v. John Tyndall, *Fragments of Science*, Vol. 1 (New York: D. Appleton, 1897), 4.

vi. *Fortnightly Review* (1865), Vol. 3, p. 129.

vii. A lighthearted quip by James Baird, a PhD student of Norman Ramsey, from a lecture at the Oak Ridge National Laboratory in 1967.

viii. President's Address to the Mathematical and Physical Section of the British Association at Norwich.

ix. Hans J. Wever and George B. Arfken, *Mathematical Methods for Physicists*, 6th ed. (Burlington, MA: Elsevier Academic Press, 2005).

x. John David Jackson, *Classical Electrodynamics*, 3rd ed. (Danvers, MA: John Wiley & Sons, 1999).

xi. Murray R. Spiegel, *Schaum's Outline Series: Advanced Mathematics for Engineers and Scientists* (McGraw-Hill, 1999).

xii. Textbooks that support practical design practices are Stephen H. Hall, Garrett W. Hall, and James A. McCall, *High-Speed Digital System Design* (New York: John Wiley & Sons, 2000), Stephen H. Hall and Howard L. Heck, *Advanced Signal Integrity for High-Speed Digital Designs* (Hoboken, NJ: John Wiley & Sons, 2009), and Howard Johnson and Martin Graham, *High-Speed Signal Propagation: Advanced Black Magic* (Upper Saddle River, NJ: Prentice Hall, 1993).

xiii. T. W. Barrett, *Topological Foundations of Electromagnetism*, World Scientific Series in Contemporary Chemical Physics, Vol. 26, 2008, begins with "Electromagnetic Phenomena Not Explained by Maxwell's Equations."

xiv. Paul G. Huray, *The Foundations of Signal Integrity* (Hoboken, NJ: John Wiley & Sons, 2009).

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Chapter 1

Foundations of Maxwell's Equations

LEARNING OBJECTIVES

- Review selected chronological developments of electromagnetic concepts
- Appreciate the role of electromagnetic theory in electrical engineering
- Use fundamental electromagnetic field quantities, units, and universal constants
- Use statistical concepts for determining the precision of a measured number
- Understand and apply principles of complex variables and phasor notation

1.1 HISTORICAL OVERVIEW

Some credit the existence of electric charge to a discovery more than two and a half thousand years ago by a Greek astronomer and philosopher, Thales of Miletus. He found that an amber (ήλεκτρον) rod, after being rubbed with silk or wool, would attract straw and small pieces of parchment. The Greek word for amber is *éléktron*, from which the words *electron*, *electronics*, *electricity*, *electromagnetic*, and *electrical engineer* are derived.

The discovery of the magnetic polarities of lodestone (μάγνηζ), a natural material found in the Thessalian Magnesia, from which we derive[i] the name *magnetic*, by Pierre de Maricourt occurred around 1269. From that time through the early seventeenth century, progress in the study of magnetism was slow, but, during the seventeenth century, there were notable contributions by a number of scientists toward understanding magnetism. A. Kirchner demonstrated that the two poles of a magnet have equal strength, and Newton attempted to formulate the laws governing the forces between bar magnets.

The inverse square law of electric and magnetic forces was not postulated until John Michell proposed it in 1750 and Coulomb confirmed it in 1785. Coulomb's

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