TABLES OF FUNCTIONS WITH FORMULAE AND CURVES

[FUNKTIONENTAFELN
MIT FORMELN UND KURVEN]

BY

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PREFACE TO 1945 EDITION

In this new edition nearly 400 corrections of errors, and other changes, have been made. Practically all of these are contributions, either directly or indirectly, of Professor R. C. ARCHIBALD, Brown University, of Doctor L. J. COMRIE, London, England, of the MATHEMATICAL TABLES PROJECT, New York City, and of Doctor J. C. P. MILLER, University of Liverpool. A supplementary bibliography compiled by Professor ARCHIBALD has been added to increase the usefulness of this work.

New York, March 1945

PREFACE TO 1938 EDITION

The principal points in which this third edition of the Tables of Functions differs from the second edition of 1933 are as follows: In the complete elliptic integrals of the first and second kind, formulae and numerical tables have been added for other than the Legendre standard forms. The numerical calculation is thereby improved in many cases. In the cylinder functions the Debye series have been brought into a more convenient form by limiting them to real values of the index and by limiting the angle of the argument to values which occur in practice; they have thus been freed from the additional determinations which necessarily added to the difficulties of the general case. Detailed numerical tables are given for the Lommel-Weber and Struve functions of orders zero and one. Formulae and graphical representations are given for the confluent hypergeometric functions and for the Mathieu functions associated with the elliptic cylinder, which will enable these two classes of functions to be employed more widely in scientific calculations.

The elementary functions which occupied the first 75 pages of the second edition have been omitted. A separate book will shortly appear in which these are given with complete numerical tables and many graphical representations. Many people who are interested in and use the elementary functions never come into contact with the higher functions and for such people the greater part of the complete work would be mere ballast.

Again I have the pleasure of thanking many helpers. Mr. H. Nagaoka (Tokio) and Mr. G. Witt (Berlin) have informed me of corrections to the tables of elliptic integrals and functions. Mr. R. Föll (now of Berlin-Siemensstadt) has calculated and drawn Fig. 28 on p. 54 for the elliptic integral of the first kind. The particulars concerning the complex zeros of the Bessel function $J_{-p}(x+iy)$ on p. 230 and 231 are taken from a thesis by Mr. W. Burkhardtsmaier. The functions $J_p(25)$ to $J_p(29)$ on p. 179 were calculated by Mr. M. J. di Toro (New York) at the suggestion of Mr. Ernst Weber.

Messrs. B. Hague and G. W. O. Howe of Glasgow kindly looked through the English text.

Mrs. E. Schopper, née Bachner, (now in Dessau) calculated the Mathieu functions and worked out the graphical representations. Mr. S. Kerridge (Stuttgart) contributed additional material.

Mr. E. Heidelbauer (Stuttgart) not only did a great amount of calculation, drawing and correction, but also undertook the difficult task of the page arrangement of the text, formulae and diagrams.

My heartfelt thanks are due to all these sincere helpers. I am indebted to the staff of Messrs. Teubner, on whose ability, care, and patience such heavy demands were made. I must also thank the publishers for meeting my wishes in such a friendly manner.

Stuttgart, April 1938.

Fritz Emde.

PREFACE TO 1933 EDITION

When in 1909 Jahnke and I published the "Functionentafeln" there was no other collection of tables of the higher functions and we therefore anticipated a great demand for our book. The sales at first by no means came up to our expectations, but since the war the book has had to be twice reprinted, and the frequent references to it in physical and technical publications show that it is much used in all parts of the world. It is again out of print. Unfortunately the preparation of this new edition was unavoidably delayed and it has taken much longer to complete than was anticipated.

It was felt that the widespread use of the tables demanded the adoption of a high standard in the preparation of the new edition, and I have done all that was in my power to make the use of the higher functions as convenient as possible. Many new tables have been added and old ones extended. If every new table had been included the price of the book would have become excessive, but where it has not been possible to include a table, curves have been given. These curves have been plotted to such a scale and with such fine subdivision that the numerical values can be read off with some accuracy.

I have tried especially to show in a clear manner by means of graphical representation the general character of the complex functions, because in this way their use is greatly simplified. The real and imaginary parts of the functions are not shown graphically because such a method of representation is completely upset if the function is multiplied by a complex constant. The function is represented by a surface the ordinates of which are equal to the modulus of the function; this surface is called the relief of the function. In this method, which is adopted here for the first time, the function reveals itself in its entirety, so that, as tests have shown, even mathematicians, who are only familiar with the function veiled as it were in formulae, hardly recognise it. Whereas formerly one had to link up in his mind the separate and disconnected characteristics and peculiarities of a function, one can now see the whole range at a glance.

Unfortunately time did not allow of all the functions being represented in this manner. Much remains to be done and I hope that Mathematical Institutes will continue the work. It is really a matter for text-books rather than for tables of functions. In every text-book of differential calculus it is regarded as essential to illustrate the nature of simple functions in the real domain by means of curves; is it considered unnecessary for more complicated functions in the complex domain?

The tables on pages 12 to 20 will shorten calculation with complex quantities, especially if one uses a slide-rule.

When we were preparing the first edition no tables existed for many of the higher functions. One had to collect what one could find. Tables for many other functions have since been calculated and published in periodicals and books, so that there are now more than one could include in a book of reasonable size intended for general use. In preparing this new edition I had therefore to pick and choose; I hope that I have chosen wisely.

As in the first edition great use has been made of the work of the Mathematical Tables Committee of the British Association. Fortunately this committee has decided to publish collections of the very accurate tables which they have calculated in past years. Two volumes have already been published. The mathematicians, physicists, and engineers of the whole world regard with the greatest wonderment and gratitude this colossal undertaking of their English colleagues, who have taken upon themselves almost entirely the heavy load of new computation. It is hardly to be conceived that other countries can continue much longer to look idly on without helping in this work.

One must not imagine that there is no need of further computation. This book, indeed, provides a general review which will enable one to judge with greater ease and certainty than heretofore just where further computation is required. It is not tables of a few functions calculated to a high degree of accuracy so much as those of many functions calculated with medium accuracy that constitute the most pressing need. Whether one wishes to become familiar with the functions or to use them it is not necessary that they be known with great accuracy. It will be time enough to compute the values to many decimal places when some exceptional practical case arises which calls for great accuracy; and even then this accuracy will, as a rule, only be required over a limited range. Even in the computation of mathematical tables the expenditure of time and trouble should bear some relation to the usefulness of the result.

As in the old edition no attempt has been made to obtain more than a reasonable degree of accuracy. The last decimal place in the values given must be regarded as uncertain. Any one who requires greater accuracy must consult tables giving a further decimal place. References to more accurate tables are given in each section.

The collections of formulae have been drastically revised with the object of making them as useful as possible to those using the tables. The formulae are intended to serve an entirely different purpose from those given in text-books and have therefore been arranged from an entirely different point of view. It must be remembered that the formulae are here merely accessory and make no claim to completeness. Some of them represent unfulfilled promises, — directions for calculating instead of the results of calculation. Some of them, on the other hand, would be necessary however many tables one had. In future editions, however, it will be necessary to cut down the collections of formulae more and more in order to make room for new tables and diagrams.

In view of the widespread use of the book in other countries it was decided at the suggestion of Dr. Alfred Giesecke-Teubner to give the explanatory text in both German and English.

With reference to various matters of detail the following points might be noted. The Table of Powers was calculated on a Brunsviga nova calculating machine in which the resulting product can with a single turn be put back into the machine as a factor for further multiplication. I must express my gratitude to Herr Otto Hess, the Stuttgart agent for the Brunsviga machines, who kindly placed the machine at my disposal. A machine with the following properties would be very useful: 1) one should be able to calculate on it from left to right without striving after an absolute accuracy which is almost always useless because the numbers with which one starts are not absolutely accurate; 2) one should be able to throw back into the machine with a single turn not only products but also quotients; 3) a storage mechanism should enable one to sum a number of terms having either all the same or alternate signs without having to write down intermediate results.

The section on cubic equations has little in common with what one finds on the subject in text-books of algebra but it gives graphical and tabular aids to their solution. Only short tables are given for the elementary transcendentals. The more extended tables, which one really requires, can be found in the books given on pages 76 to 78. The real subject of this book — the higher functions — begins on page 78. Spherical harmonics of the second kind and the associated spherical harmonics have been added to the section on spherical harmonics and are represented by curves. The section on Bessel functions has been greatly extended. The Nielsen definition of the functions of the second and third kind (N and H) is now uniformly introduced throughout all the tables. The multiplicity of definitions which was formerly so troublesome has thus been done away with. The tables of the more important Bessel functions are now so complete that they can be used as conveniently as the circular functions. Riemann's Zeta function has been included. Many aids and simplifications are at the service of those who have to carry out calculations if they know of their existence and where to look for them; the bibliography at the end of the book will be found useful in indicating where such aids are to be found.

I am indebted to Herr Karl Willy Wagner for the curves on pages 239 to 241 showing the behaviour of the Bessel function $J_p(x)$ for a real argument x and high order p. These curves were calculated in 1922 by Herr Hans Salinger and Herr Hans Stahl of the Technical Department of the Imperial Telegraphs. Those functions related to $J_p(x)$ which I have called $\Lambda_p(x)$ were calculated by Herr A. Walther (Darmstadt), Herr S. Gradstein and Herr K. Hessenberg at the suggestion of Herr R. Straubel of the Zeiss Works in Jena. The tables (p. 250—258) appear here for the first time.

My thanks are due to all those above mentioned and also to the many friends, too numerous to mention by name, in all parts of the world, who have in the past been good enough to write to Jahnke or to me making corrections and suggestions.

Eugen Jahnke died of heart failure on 18th October 1921 at the age of 57. Only a few weeks previously I had discussed with him at Rudolstadt the programme for the new edition. Neither of us at that time had any idea what it would ultimately look like. He was not able to take any further part in the work.

My wife, who had assisted in the preparation of the first edition, also did much laborious computation for the new edition and it was a grief to her when increasing ill-health compelled her to give up this work. Fate denied her the pleasure of seeing the completion of the book.

I am deeply indebted to Professor G. W. O. Howe of Glasgow University who has read and corrected the English text — a friendly token of the days now long past when we were associated in the electrical industry.

It would have been impossible for me to have made such far-reaching improvements in the new edition had it not been for the long continued and able assistance of my fellow worker Herr Rudolf Rühle. He has made most of the new calculations, drawn all the new diagrams, collected the formulae for the spherical harmonics and the Riemann Zeta Function and translated the text into English. The greater part of the task of correction has also fallen to him. The arrangement of the tables, diagrams and formulae with respect to the pages was often a difficult matter requiring considerable ingenuity. This new edition thus owes very much to Herr Rühle and I am deeply indebted to him. I must also thank Herr Dipl.-Ing. Gottfried Hänsch for help with the computation and Herr Erich Heidelbauer who has given great assistance with the drawings and other matters.

Finally my sincere thanks are due to Mess^rs^ Teubner and their staff, especially Herr Thilo, for their patience under trying circumstances. A special word of praise is due to the publishers for their efforts to keep the price of the book as low as possible. That the quality of the production has not been allowed to suffer thereby is obvious from an examination of the book.

Stuttgart, August 1933.

Fritz Emde.

Figurenverzeichnis. | Index of figures.

Abkürzung: | Abbreviation:

P. R. = Projektion des Reliefs = Höhenkarte = Altitude chart.

Figurenverzeichnis.
Index of figures.

Durchgängig gebrauchte Abkürzungen. | Abbreviations used throughout.

$$i^{\varrho} = \cos \varrho^{\llcorner} + i \sin \varrho^{\llcorner} \equiv e^{i\frac{\pi}{2}\varrho} = \cos \frac{\pi}{2}\varrho + i \sin \frac{\pi}{2}\varrho.$$

Winkeleinheiten: / Angular units: $\frac{\pi}{2}$ rad $= 90^\circ = 1^{\llcorner}$ (lies: 1 Rechter / read: 1 right angle)

${}^{n}4 \equiv 0{,}4 \cdot 10^{n}$; $\quad 2^{n}\,9 \equiv 2{,}9 \cdot 10^{n}$ $\quad 3041 \mid n = 3{,}041 \cdot 10^{n}$.

$0{,}0^{3}\,7 \equiv 0.000\,7 = 0\,7 \cdot 10^{-3}$.

ADDENDA

(Pages referred to below will be found in Addenda, following General Index on pages 302-303.)

Contents.

The obliquely printed page numbers refer to the tables.

Figurenverzeichnis. | Index of figures.

Abkürzung: | Abbreviation:

P. R. = Projektion des Reliefs = Höhenkarte = Altitude chart.

Contents.

The obliquely printed page numbers refer to the tables.

I. Integral-Sinus, -Kosinus und -Logarithmus.
I. Sine, cosine and logarithmic integral.

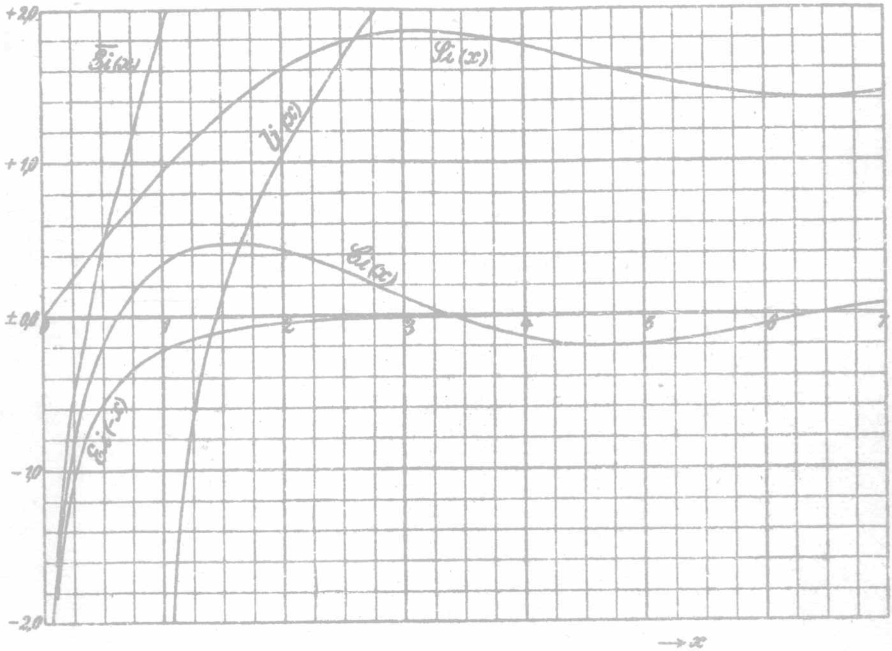

Fig. 1. Integral-Sinus, -Kosinus und -Logarithmus.
Fig. 1. Sine, cosine and logarithmic integral

1. Definitionen.
1. Definitions.

$$-\operatorname{Ei}(-x) = \int_{-\infty}^{-x} \frac{e^t}{-t}\, dt = \int_{x}^{\infty} \frac{e^{-t}}{t}\, dt > 0, \qquad \infty > x > 0.$$

$$\operatorname{Ei}(-x e^{i\pi}) = \operatorname{Ei}(x - i0) = \operatorname{Ei}^-(x)$$

$$\operatorname{Ei}(-x e^{-i\pi}) = \operatorname{Ei}(x + i0) = \operatorname{Ei}^+(x)$$

$$\mathfrak{S}\mathrm{i}\, x = \int_0^x \frac{\mathfrak{S}\mathrm{in}\, t}{t}\, dt = \tfrac{1}{2}\int_{-x}^{+x} \frac{\mathfrak{S}\mathrm{in}\, t}{t}\, dt = \tfrac{1}{2} \lim_{\varepsilon \to 0} \left(\int_{-x}^{-\varepsilon} \frac{e^t}{t}\, dt + \int_{\varepsilon}^{x} \frac{e^t}{t}\, dt \right)$$

$$= x + \frac{x^3}{3!\,3} + \frac{x^5}{5!\,5} + \cdots$$

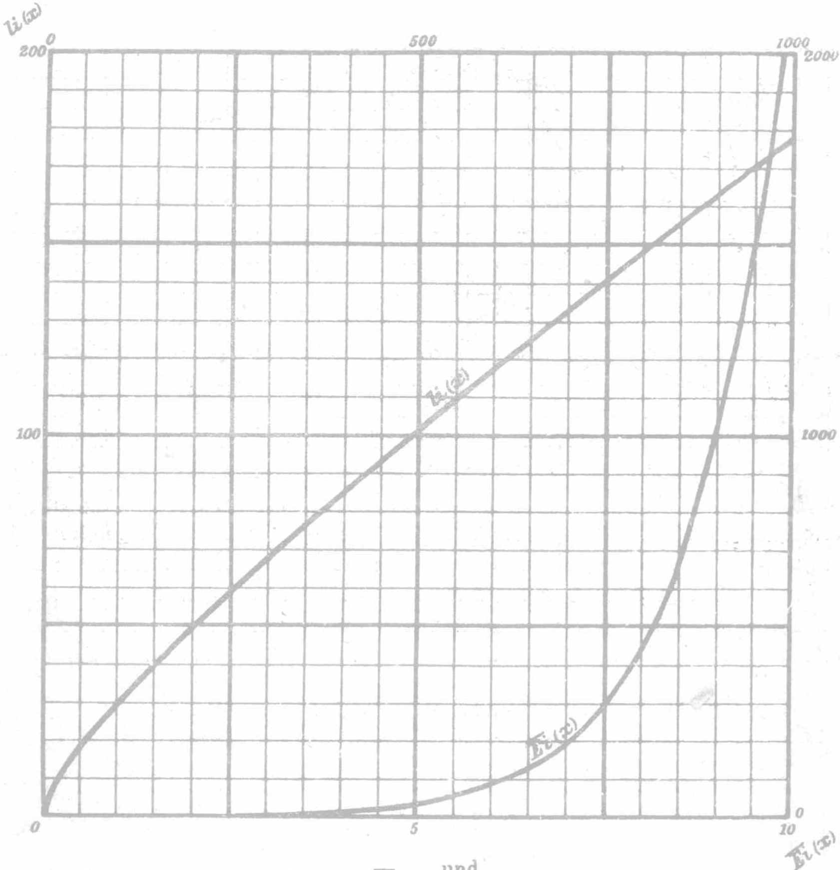

Fig. 2. $\overline{\mathrm{Ei}}\,(x)$ und / and $\mathrm{li}\,(x)$.

$$\mathfrak{Ci}\,x = \mathrm{Ei}\,(-x) + \mathfrak{Si}\,x = \ln \gamma x + \int_0^x \frac{\mathfrak{Cos}\,t - 1}{t}\,dt = \ln \gamma x + \frac{x^2}{2!\,2} + \frac{x^4}{4!\,4} + \cdots,$$

wo | where

$$\ln \gamma = C = \int_0^1 \frac{1 - e^{-t}}{t}\,dt - \int_1^\infty \frac{e^{-t}}{t}\,dt = 0{,}796\,599\,599 - 0{,}219\,383\,934$$

$$= 0{,}577\,215\,665,$$

$$\gamma = e^C = 1{,}781\,072.$$

$$\mathrm{Ei}\,x = \mathfrak{Ci}\,x + \mathfrak{Si}\,x = \mathrm{li}\,e^x$$ (exponential integral)

$$= \mathrm{Ei}^+(x) + i\pi = \mathrm{Ei}^-(x) - i\pi = \tfrac{1}{2}\,\mathrm{Ei}^+(x) + \tfrac{1}{2}\,\mathrm{Ei}^-(x).$$

$$\text{li}\, x = \int_0^x \frac{dt}{\ln t} = \overline{\text{Ei}}\,(\ln x)$$ (Integral-Logarithmus) (Logarithmic-integral)

$$\mathfrak{Si}\, iy = i \cdot \text{Si}\, y, \qquad \text{Si}\, x = \int_0^x \frac{\sin t}{t}\, dt, \qquad \text{Si}\, \infty = \frac{\pi}{2}$$

$$\text{Si}\, iy = i \cdot \mathfrak{Si}\, y, \qquad \text{Si}\, x = \frac{\pi}{2} + \text{si}\, x, \qquad \text{si}\, x = -\int_x^\infty \frac{\sin t}{t}\, dt$$

$$\mathfrak{Ci}\, iy = i\frac{\pi}{2} + \text{Ci}\, y, \qquad \text{Ci}\, iy = i\frac{\pi}{2} + \mathfrak{Ci}\, y,$$

$$\text{Ci}\, x = -\int_x^\infty \frac{\cos t}{t}\, dt = \ln \gamma x - \int_0^x \frac{1-\cos t}{t}\, dt$$

(Integral-Kosinus | Cosine integral)

$\left.\begin{matrix}\text{Si}\, x \\ \text{si}\, x\end{matrix}\right\}$ (Integral-Sinus | Sine integral)

Näherungswerte bei kleinem x: | Approximations for small values of x:

$$\mathfrak{Si}\, x \approx \text{Si}\, x \approx x, \qquad \text{si}\, x \approx -\left(\frac{\pi}{2} - x\right)$$

$$\mathfrak{Ci}\, x \approx \text{Ci}\, x \approx \overline{\text{Ei}}\, x \approx \text{Ei}\,(-x) \approx -\ln \frac{1}{\gamma x}, \qquad \text{li}\, x \approx -\frac{x}{\ln \frac{1}{x}}$$

2. Halbkonvergente Reihe.
2. Semi-convergent series.

$$\overline{\text{Ei}}\, x = \frac{e^x}{x} H(x), \qquad H(x) = 1 + \frac{1!}{x} + \frac{2!}{x^2} + \cdots, \qquad |x| >> 1.$$

$$-\text{si}\, y + i \cdot \text{Ci}\, y = \frac{e^{iy}}{y} H(iy)$$

$$\text{Ci}\, x \approx +\frac{\sin x}{x}, \qquad \text{si}\, x \approx -\frac{\cos x}{x}, \qquad \text{Si}\, x \approx \frac{\pi}{2} - \frac{\cos x}{x}, \qquad x >> 1.$$

3. Negative und rein imaginäre Argumente.
3. Negative and pure imaginary arguments.

$$\mathfrak{Si}(-x) = -\mathfrak{Si}\, x, \quad \mathfrak{Ci}\,(x e^{+i\pi}) = \mathfrak{Ci}\, x \pm i\pi, \quad \overline{\text{Ei}}\,(x e^{\pm i\pi}) = \text{Ei}\,(-x) \pm i\pi$$

$$\text{Si}\,(-x) = -\text{Si}\, x, \quad \text{si}\, x + \text{si}\,(-x) = -\pi, \quad \text{Ci}\,(x e^{\pm i\pi}) = \text{Ci}\, x \pm i\pi.$$

$$\text{Ci}\, y \pm i\, \text{Si}\, y = \overline{\text{Ei}}\,(\pm iy) \mp i\frac{\pi}{2}$$

$$\text{Ci}\, y + i\, \text{si}\, y = \overline{\text{Ei}}\, iy - i\pi \quad = \text{Ei}^+(iy)$$

$$\text{Ci}\, y - i\, \text{si}\, y = \overline{\text{Ei}}\,(-iy) + i\pi = \text{Ei}^-(-iy).$$

4. Integrale.

4. Integrals.

$$\int_x^\infty \frac{e^{-mt}}{a+t}\,dt = -e^{ma}\,\mathrm{Ei}\,[-m(a+x)]$$

$$\int_x^\infty \frac{e^{im(b+t)}}{a+t}\,dt = -e^{im(b-a)}\,\mathrm{Ei}^+\,im(a+x)$$

$$\int_0^\infty \frac{t-ia}{t^2+a^2}\,e^{imt}\,dt = -e^{ma}\,\mathrm{Ei}\,(-ma)$$

$$\int_0^\infty \frac{t+ia}{t^2+a^2}\,e^{imt}\,dt = -e^{-ma}\,\mathrm{Ei}^+\,ma$$

$$\int_0^x \mathrm{Ei}\,(-mt)\,dt = x\,\mathrm{Ei}\,(-mx) - \frac{1-e^{-mx}}{m}$$

$$\int_0^x \mathrm{Ei}^+\,(imt)\,dt = x\,\mathrm{Ei}^+\,imx + \frac{1-e^{imx}}{im}$$

$$\int_0^\infty e^{-pt}\,\mathrm{Ci}\,qt\,dt = -\frac{1}{p}\ln\left(1+\frac{p^2}{q^2}\right)$$

$$\int_0^\infty e^{-pt}\,\mathrm{si}\,qt\,dt = -\frac{1}{p}\,\mathrm{arc\,tg}\,\frac{p}{q}$$

$$\int_0^\infty \cos t\,\mathrm{Ci}\,t\,dt = \int_0^\infty \sin t\,\mathrm{si}\,t\,dt = -\frac{\pi}{4}$$

$$\int_0^\infty \mathrm{Ci}^2\,t\,dt = \int_0^\infty \mathrm{si}^2\,t\,dt = \frac{\pi}{2}, \qquad \int_0^\infty \mathrm{Ci}\,t\,\mathrm{si}\,t\,dt = -\ln 2.$$

5. Sici-Spirale.

Trägt man $\mathrm{Ci}\,x$ und $\mathrm{Si}\,x$ als rechtwinklige Koordinaten einer ebenen Kurve auf (Fig. 3), so ist der Krümmungsradius der Kurve $R=\frac{1}{x}=e^{-s}$ ($s=$ Bogenlänge). Die Krümmung wächst exponentiell mit der Bogenlänge. Solche Spiralen eignen sich als Profile von Kurvenlinealen.

5. Sici spiral.

If we draw $\mathrm{Ci}\,x$ and $\mathrm{Si}\,x$ as rectangular coordinates of a plane curve (fig. 3), then the radius of curvature of the curve is $R=\frac{1}{x}=e^{-s}$ ($s=$ length of arc). The curvature increases exponentially with the length of arc. Such spirals are suitable as profiles of French curves.