# Mathematical Programs with Equilibrium Constraints

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This book provides a solid foundation and an extensive study for an important class of constrained optimization problems known as Mathematical Programs with Equilibrium Constraints (MPEC), which are extensions of bilevel optimization problems. The book begins with the description of many source problems arising from engineering and economics that are amenable to treatment by the MPEC methodology. Error bounds and parametric analysis are the main tools to establish a theory of exact penalization, a set of MPEC constraint qualifications and the first- and second-order optimality conditions. The book also describes several iterative algorithms such as a penalty-based interior point algorithm, an implicit programming algorithm and a piecewise sequential quadratic programming algorithm for MPECs. Results in the book will have significant impacts in such disciplines as engineering design, economics and game equilibria, and transportation planning, within all of which MPEC has a central role to play in the modeling of many practical problems.

A useful resource for applied mathematicians in general, this book will be a particularly valuable tool for operations researchers, transportation, industrial, and mechanical engineers, and mathematical programmers.

**Mathematical Programs
with
Equilibrium Constraints**

To our families

# Numbering System

The six chapters of the book are numbered from 1 to 6, the sections are denoted by decimal numbers of the type **2.3** (meaning Section 3 of Chapter 2). Many sections are further divided into subsections, some subsections are numbered, others are not. The numbered subsections are by decimal numbers following the section numbers; e.g., Subsection **1.3.1** means Chapter 1, Section 3, Subsection 1.

All definitions, results, and miscellaneous items are numbered consecutively within each section in the form **1.3.5**, **1.3.6**, meaning Items 5 and 6 in Section 3 of Chapter 1. All items are also identified by their types (e.g., **1.4.1 Proposition.**, **1.4.2 Remark.**). When an item is referred to in the text, it is called out as Algorithm **5.2.1**, Theorem **4.1.7**, etc.

Equations are numbered consecutively in each section by (1), (2), etc. Any reference to an equation in the same section is by this number only, whereas equations in another section are identified by chapter, section, and equation. Thus (3.1.4) means Equation (4) in Section 1 of Chapter 3.

# Acronyms

| | |
|---|---|
| AVI | Affine Variational Inequality |
| BIF | B(ouligand)-Differentiable Implicit Function Condition |
| CQ | Constraint Qualification |
| $C^r$ | Continuously differentiable of order $r$ |
| CRCQ | Constant Rank Constraint Qualification |
| GBIF | Global BIF Condition |
| IMP | Implicit Programming |
| KKT | Karush-Kuhn-Tucker |
| LCP | Linear Complementarity Problem |
| LICQ | Linear Independence Constraint Qualification |
| MFCQ | Mangasarian-Fromovitz Constraint Qualification |
| MP | Mathematical Program |
| MPAEC | Mathematical Program with Affine Equilibrium Constraints |
| MPEC | Mathematical Program with Equilibrium Constraints |
| NCP | Nonlinear Complementarity Problem |
| NLP | Nonlinear Program |
| PCP | Piecewise Programming |
| $PC^r$ | Piecewise smooth of order $r$ |
| PIPA | Penalty Interior Point Algorithm |
| PSQP | Piecewise Sequential Quadratic Programming |
| SBCQ | Sequentially Bounded Constraint Qualification |
| SCOC | Strong Coherent Orientation Condition |
| SMFCQ | Strict Mangasarian-Fromovitz Constraint Qualification |
| SQP | Sequential Quadratic Programming |
| SRC | Strong Regularity Condition |
| VI | Variational Inequality |

# Glossary of Notation

## Scalars

| | |
|---|---|
| sgn $t$ | the sign, $1, -1, 0$, of a positive, negative, or zero scalar $t$ |
| $t_+ \equiv \max(0, t)$ | the nonnegative part of a scalar |
| $t_- \equiv \max(0, -t)$ | the nonpositive part of a scalar |

## Spaces

| | |
|---|---|
| $\Re^n$ | real $n$-dimensional space |
| $\Re$ | the real line |
| $\Re^{n \times m}$ | the space of $n \times m$ real matrices |
| $\Re^n_+$ | the nonnegative orthant of $\Re^n$ |
| $\Re^n_{++}$ | the positive orthant of $\Re^n$ |

## Vectors

| | |
|---|---|
| $z^T$ | the transpose of a vector $z$ |
| $\{z^k\}$ | a sequence of vectors $z^1, z^2, z^3, \ldots$ |
| $x^T y$ | the standard inner product of vectors in $\Re^n$ |
| $\|x\| \equiv \sqrt{x^T x}$ | the Euclidean norm of a vector $x \in \Re^n$ |
| $x \geq y$ | the (usual) partial ordering: $x_i \geq y_i, i = 1, \ldots n$ |
| $x > y$ | the strict ordering: $x_i > y_i, i = 1, \ldots, n$ |
| $\min(x, y)$ | the vector whose $i$-th component is $\min(x_i, y_i)$ |
| $\max(x, y)$ | the vector whose $i$-th component is $\max(x_i, y_i)$ |
| $x \circ y \equiv (x_i y_i)$ | the Hadamard product of $x$ and $y$ |
| $x \perp y$ | $x$ and $y$ are perpendicular |
| $z^+ \equiv \max(0, z)$ | the nonnegative part of a vector $z$ |
| $z^- \equiv \max(0, -z)$ | the nonpositive part of a vector $z$ |

## Matrices

| | |
|---|---|
| $\det A$ | the determinant of a matrix $A$ |
| $A^{-1}$ | the inverse of a matrix $A$ |
| $\|A\|$ | the Euclidean norm of a matrix $A$ |
| $A^T$ | the transpose of a matrix $A$ |
| $A_\alpha$ | the columns of $A$ indexed by $\alpha$ |
| $A_{\alpha\cdot}$ | the rows of $A$ indexed by $\alpha$ |
| $I$ | the identity matrix of appropriate order |
| $I_k$ | the identity matrix of order $k$ |
| $\mathrm{diag}(a)$ | the diagonal matrix with diagonal elements equal to the components of the vector $a$ |

## Functions

| | |
|---|---|
| $f : \mathcal{D} \to \mathcal{R}$ | a mapping with domain $\mathcal{D}$ and range $\mathcal{R}$ |
| $f \circ g$ | composition of two functions $f$ and $g$ |
| $\nabla f$ | $(\partial f_i / \partial x_j)$, the $m \times n$ Jacobian of a mapping $f : \Re^n \to \Re^m$ $(m \geq 2)$ |
| $\nabla_\beta f_\alpha$ | $(\partial f_i / \partial x_j)_{i \in \alpha}^{j \in \beta}$, a submatrix of $\nabla f$ |
| $\nabla \theta$ | $(\partial \theta / \partial x_j)$, the gradient of a function $\theta : \Re^n \to \Re$ |
| $\nabla_y g(x, y)$ | the partial Jacobian matrix of $g$ with respect to $y$ |
| $\nabla^2 \theta$ | Hessian matrix of the scalar-valued function $\theta$ |
| $f'(\cdot\,; \cdot)$ | directional derivative of the mapping $f$ |
| $f^{-1}$ | the inverse of $f$ |
| $o(t)$ | any function such that $\lim_{t \to 0} \frac{o(t)}{t} = 0$ |
| $\Pi_K(x)$ | the Euclidean projection of $x$ on the set $K$ |
| $\inf f(x)$ | the infimum of the function $f$ |
| $\sup f(x)$ | the supremum of the function $f$ |
| $\mathrm{dist}(x, W)$ | distance function from vector $x$ to set $W$ |
| $F_C$ | normal map associated with function $F$ and set $C$ |

## Sets

| | |
|---|---|
| $\in$ | element membership |
| $\notin$ | not an element of |
| $\emptyset$ | the empty set |
| $\subseteq$ | set inclusion |
| $\subset$ | proper set inclusion |
| $\cup, \cap, \times$ | union, intersection, Cartesian product |
| $\prod S_i$ | Cartesian product of sets $S_i$ |
| $S_1 \setminus S_2$ | the difference of sets $S_1$ and $S_2$ |
| $\lvert S \rvert$ | the cardinality of a finite set $S$ |
| $\partial S$ | the (topological) boundary of a set $S$ |
| $\mathrm{cl}\, S$ | the (topological) closure of a set $S$ |
| $S^*$ | the dual cone of $S$ |
| $0^+ S$ | the cone of recession directions of $S$ |
| $\mathrm{Gr}(\mathcal{A})$ | the graph of a multifunction $\mathcal{A}$ |
| $\mathrm{dom}(\mathcal{A})$ | the domain of a multifunction $\mathcal{A}$ |
| $\mathbb{B}(x, \delta)$ | the closed ball with center at $x$ with radius $\delta$ |
| $\mathrm{argmin}_x f(x)$ | the set of $x$ attaining the minimum of the real-valued function $f(x)$ |
| $\mathrm{argmax}_x f(x)$ | the set of $x$ attaining the maximum of the real-valued function $f(x)$ |
| $\mathrm{supp}(x)$ | the support of vector $x$ |
| $\mathcal{T}(x; S)$ | tangent cone of set $S$ at point $x \in S$ |
| $\mathcal{C}(x; S)$ | critical cone of set $S$ at point $x \in S$ relative to an objective function |
| $\mathcal{N}(x; S)$ | normal cone of set $S$ at point $x \in S$ |
| $[a, b]$ | a closed interval in $\Re$ |
| $(a, b)$ | an open interval in $\Re$ |
| $x^\perp$ | the orthogonal complement of vector $x$ |

## Problems

| | |
|---|---|
| AVI $(q, M, K)$ | AVI defined by vector $q$, matrix $M$ and set $K$ |
| LCP $(q, M)$ | LCP defined by vector $q$ and matrix $M$ |
| $\mathrm{SOL}(F, K)$ | solution set of the VI $(F, K)$ |
| $\mathrm{SOL}(q, M, K)$ | solution set of the AVI $(q, M, K)$ |
| VI $(F, K)$ | VI defined by mapping $F$ and set $K$ |

**MPEC symbols**

| | |
|---|---|
| $v \equiv (\zeta, \pi, \eta)$ | MPEC multipliers |
| $w \equiv (x, y, \lambda)$ | variable of MPEC in KKT form, $\lambda \in M(x, y)$ |
| $y(x)$ | implicit solution function of lower-level VI |
| $z \equiv (x, y)$ | original MPEC variable |
| $\mathcal{B}(\bar{x})$ | SCOC family of active index sets that define the $C^1$ pieces of $y(x)$ at $\bar{x}$ |
| $\mathcal{C}(z; \mathcal{F})$ | critical cone of MPEC at $z \in \mathcal{F}$ relative to the objective function $f$ $\equiv \bigcup_{\lambda \in M(z)} \mathcal{C}(z, \lambda)$ under full MPEC CQ |
| $\mathcal{C}(z, \lambda)$ | a piece of $\mathcal{C}(z; \mathcal{F})$ corresponding to $\lambda \in M(z)$ $\equiv \mathcal{T}(z; Z) \cap \text{Gr}(\mathcal{LS}_{(z,\lambda)}) \cap \nabla f(z)^{\perp}$ |
| $\mathcal{F}$ | MPEC's feasible region given by $Z \cap \text{Gr}(\mathcal{S})$ |
| $\mathcal{F}^{\text{KKT}}$ | feasible region of MPEC in KKT form |
| $\mathcal{I}(x, y)$ | set of active indices at $(x, y) \in \text{Gr}(\mathcal{S})$ $\equiv \{i : g_i(x, y) = 0\}$ |
| $\mathcal{I}_0(x, y, \lambda)$ | degenerate index set at $(x, y, \lambda) \in \mathcal{F}^{\text{KKT}}$ $\equiv \{i : \lambda_i = g_i(x, y) = 0\}$ |
| $\mathcal{I}_+(x, y, \lambda)$ | nondegenerate index set at $(x, y, \lambda) \in \mathcal{F}^{\text{KKT}}$ $\equiv \{i : \lambda_i > g_i(x, y) = 0\}$ |
| $\mathcal{K}(z, \lambda)$ | lifted critical cone at $(z, \lambda) \in \mathcal{F}^{\text{KKT}}$ |
| $\mathcal{K}(z, \lambda; dx)$ | directional critical set at $(z, \lambda) \in \mathcal{F}^{\text{KKT}}$ along direction $dx$ |
| $L(x, y, \lambda)$ | Lagrangean function for lower-level VI $\equiv F(x, y) + \sum_{i=1}^{\ell} \lambda_i \nabla_y g_i(x, y)$ |
| $\mathcal{L}(z; \mathcal{F})$ | MPEC linearized cone at $z \in \mathcal{F}$ $\equiv \mathcal{T}(z, Z) \cap \left( \bigcup_{\lambda \in M(z)} \text{Gr}(\mathcal{LS}_{(z,\lambda)}) \right)$ |
| $\mathcal{L}^{\text{MPEC}}(w, \zeta, \pi, \eta)$ | MPEC Lagrangean function |
| $\mathcal{LS}_{(z,\lambda)}$ | linearized solution map at $(z, \lambda) \in \mathcal{F}^{\text{KKT}}$ for lower-level VI; $\mathcal{LS}_{(z,\lambda)}(dx)$ is defined as $\text{SOL}(\nabla_x L(z, \lambda) dx, \nabla_y L(z, \lambda), \mathcal{K}(z, \lambda, dx))$ |
| $\mathcal{LS}^{\text{KKT}}_{(z,\lambda)}(dx)$ | set of KKT pairs $(dy, d\lambda)$ of the AVI $(\nabla_x L(z, \lambda) dx, \nabla_y L(z, \lambda), \mathcal{K}(z, \lambda; dx))$ |

**MPEC symbols**
(continued)

| | |
|---|---|
| $M(x,y)$ | set of KKT multipliers of VI $(F(x,\cdot),C(x))$ at solution $y$ |
| $M^c(z;dx)$ | set of critical multipliers at $z \in \mathcal{F}$ along direction $dx$ $\equiv \{\lambda \in M(z) : \mathcal{K}(z,\lambda;dx) \neq \emptyset\}$ |
| $M^e(x,y)$ | set of extreme points of $M(x,y)$ |
| $\mathcal{S}(x)$ | set of rational reactions of lower-level VI $\equiv \mathrm{SOL}(F(x,\cdot),C(x))$ |
| $Z$ | upper-level feasible region of $(x,y)$ |

# Preface

This monograph deals with a class of constrained optimization problems which we call *Mathematical Programs with Equilibrium Constraints*, or simply, MPECs. Briefly, an MPEC is an optimization problem in which the essential constraints are defined by a parametric variational inequality or complementarity system. The terminology, MPEC, is believed to have been coined in [108]; the word "equilibrium" is adopted because the variational inequality constraints of the MPEC typically model certain equilibrium phenomena that arise from engineering and economic applications. The class of MPECs is an extension of the class of *bilevel programs*, also known as mathematical programs with optimization constraints, which was introduced in the operations research literature in the early 1970s by Bracken and McGill in a series of papers [34, 36, 37]. The MPEC is closely related to the economic problem of Stackelberg game [265] the origin of which predates the work of Bracken and McGill.

Our motivation for writing this monograph on MPEC stems from the practical significance of this class of mathematical programs and the lack of a solid basis for the treatment of these problems. Although there is a substantial amount of previous research on special cases of MPEC, no existing work provides such generality, depth, and rigor as the present study. Our intention in this monograph is to establish a sound foundation for MPEC that we hope will inspire further applications and research on this important problem.

This monograph consists of six chapters. Chapter **1** defines the MPEC, gives a brief description of several source problems, and presents various equivalent formulations of the equilibrium constraints in MPEC; the chapter concludes with some results of existence of optimal solutions. Chapter **2** presents an extensive theory of exact penalty functions for MPEC, using the theory of error bounds for inequality systems. This chapter ends with a brief discussion of how some exact penalty functions formulations of MPEC can be employed to obtain first-order optimality conditions; the latter topic and its extensions are treated in full in the next three chapters. Specifically, Chapter **3** presents the fundamental first-order optimality (i.e., stationarity) conditions of MPEC; Chapter **4** verifies in detail the hypotheses needed for the first-order conditions; Chapter **5** contains results on second-order optimality conditions. The sixth and last chapter presents several algorithms for solving MPECs including an interior point algorithm for MPECs with "monotone" inner problems, a conceptual iterative descent algorithm based on an implicit programming approach, and a locally superlinearly convergent Newton type (sequential quadratic programming) method based on a piecewise programming approach. Some preliminary computational results are reported. The monograph ends with an extensive list of references.

Due to the intrinsic complexity of the MPEC, a comprehensive study of this problem would inevitably require extensive tools from diverse disciplines. Besides a general knowledge of smooth (nonlinear) programming and multivariate analysis, which we assume as prerequisites for this work, such subjects as error bound theory for inequality systems, sensitivity and stability theory for parametric variational inequalities, piecewise smooth analysis, nonsmooth equations, the family of interior point methods, and some basic iterative descent methods for nonlinear programs are all important tools that will be used in this monograph. Since it is not possible for us to review in detail all the background material and keep the monograph within a reasonable length, we have chosen not to organize the preliminary results separately. Instead, we have included only the most useful background results relevant to the topics of discussion.

Throughout the monograph, we have taken several different points of view toward the MPEC, each of which is interesting by itself. Many results obtained herein are new and have not appeared in the literature before. For related approaches and results, we refer to [201, 214, 291, 292, 295]; see also the references in [5, 278].

The general MPEC is a highly nonconvex, nondifferentiable optimization problem that encompasses certain combinatorial features in its constraints. As such, it is computationally very difficult to solve, especially if one wishes to compute a globally optimal solution. Partly due to this pessimistic view, we have not attempted in this monograph to deal with the issue of finding a globally optimal solution to the general problem itself or to its special cases. The algorithms discussed in Chapter **6** are iterative schemes for computing a stationary point of the MPEC (and under mild conditions, a strict local minimum). We refer to [278] for references that discuss some global optimization approaches to solving bilevel programs.

Due to the broad applications of MPEC, this monograph is of interest to readers from diverse disciplines. In particular, operations researchers, economists, design and systems engineers, and applied mathematicians will likely find the subject matter interesting and challenging. We have written the monograph with these individuals in mind.

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