Second Edition

Elementary Differential Equations with Linear Algebra

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ELEMENTARY DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA

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Second Edition

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Preface

The purpose of this book is to provide an introduction to differential equations and linear algebra that takes advantage of the interplay between the two subjects. Linear and matrix algebra are useful tools for both computational and theoretical work in differential equations. This is particularly true in the study of linear systems of differential equations. On the other hand, differential equations provide examples and applications of many concepts in linear algebra.

The level of the book is such that it is accessible to students who have taken two or three terms of calculus. Applications have been presented from a variety of fields. Proofs of several of the more difficult theorems have been omitted, but most results have been proved. The theorems in the text, along with the more challenging exercises, can be used to introduce the student to mathematical rigor at an elementary level.

Most of the material on linear algebra is in Chapters II, III, and IV. Section 3.6 can be omitted without loss of continuity. Chapter IV (Characteristic Values) can be omitted if the characteristic value method for solving systems of differential equations in Chapter VI (Sections 6.4–6.7) is also omitted. Chapter V can be taken up before Chapter IV if desired. A choice from the applications can be made, depending on the interests of the students.

The second edition contains additional applications. These include population growth, economic dynamics, the two-body problem, stochastic matrices, and applications for linear systems of algebraic equations. Much of the material on linear algebra has been rewritten. Fewer topics are presented and the treatment of the remaining topics has been expanded with the intention of making the level more elementary. Specific changes include the following. Systems of linear equations are solved by reducing them to row–echelon form. The discussion of linear dependence has been simplified. Additional exercises and examples have been inserted to illustrate basic ideas such as subspaces, linear independence, orthogonal bases, and linear transformations. Some of the more theoretical material has been dropped. Other minor changes have been made where it was felt that the presentation could be improved.

Answers to about half the computational problems are given at the end of the book.

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I

Introduction to Differential Equations

1.1 INTRODUCTION

An ordinary differential equation may be defined as an equation that involves a single unknown function of a single variable and some finite number of its derivatives. For example, a simple problem from calculus is that of finding all functions f for which

$$f'(x) = 3x^2 - 4x + 5 \tag{1.1}$$

for all x. Clearly a function f satisfies the condition (1.1) if and only if it is of the form

$$f(x) = x^3 - 2x^2 + 5x + c,$$

where c is an arbitrary number. A more difficult problem is that of finding all functions g for which

$$g'(x) + 2[g(x)]^2 = 3x^2 - 4x + 5. \tag{1.2}$$

Another difficult problem is that of finding all functions y for which (we use the abbreviation y for $y(x)$)

$$x^2 \frac{d^2y}{dx^2} - 3x\left(\frac{dy}{dx}\right)^2 + 4y = \sin x. \tag{1.3}$$

In each of the problems (1.1), (1.2), and (1.3) we are asked to find all functions that satisfy a certain condition, where the condition involves one or more *derivatives* of the function. We can reformulate our definition of a differential equation as follows. Let F be a function of $n + 2$ variables. Then the equation

$$F[x, y, y', y'', \ldots, y^{(n)}] = 0 \tag{1.4}$$

is called an ordinary differential equation of order n for the unknown function y. The *order* of the equation is the order of the highest order derivative that appears in the equation. Thus, Eqs. (1.1) and (1.2) are first-order equations, while Eq. (1.3) is of second order.

A *partial* differential equation (as distinguished from an *ordinary* differential equation) is an equation that involves an unknown function of more than one independent variable, together with partial derivatives of the function. An example of a partial differential equation for an unknown function $u(x, t)$ of two variables is

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + u .$$

Almost all the differential equations that we shall consider will be ordinary.

By a *solution* of an ordinary differential equation of order n, we mean a function that, on some interval,[1] possesses at least n derivatives and satisfies the equation. For example, a solution of the equation

$$\frac{dy}{dx} - 2y = 6$$

is given by the formula

$$y = e^{2x} - 3, \qquad \text{for all } x,$$

because

$$\frac{d}{dx}(e^{2x} - 3) - 2(e^{2x} - 3) = 2e^{2x} - 2e^{2x} + 6 = 6$$

[1] We shall use the notations (a, b), $[a, b]$, $(a, b]$, $[a, b)$, (a, ∞), $[a. \infty)$, $(-\infty, a)$, $(-\infty, a]$, $(-\infty, \infty)$ for intervals. Here (a, b) is the set of all real numbers x such that $a < x < b$, $[a, b]$ is the set of all real numbers x such that $a \leq x \leq b$, $[a, b)$ is the set of all real numbers x such that $a \leq x < b$, and so on.

for all x. The set of all solutions of a differential equation is called the *general solution* of the equation. For instance, the general solution of the equation

$$\frac{dy}{dx} = 3x^2 - 4x$$

consists of all functions that are of the form

$$y = x^3 - 2x^2 + c, \qquad x \text{ in } \mathscr{I},$$

where c is an arbitrary constant and $\mathscr{I}$ is an arbitrary interval. To *solve* a differential equation is to find its general solution.

Let us now solve the second-order equation

$$\frac{d^2y}{dx^2} = 12x + 8.$$

Integrating, we find that

$$\frac{dy}{dx} = 6x^2 + 8x + c_1,$$

where c_1 is an arbitrary constant. A second integration yields

$$y = 2x^3 + 4x^2 + c_1x + c_2$$

for the general solution. Here c_2 is a second arbitrary constant.

The general solution of the third-order equation

$$y''' = 16e^{-2x}$$

can be found by three successive integrations. We find easily that

$$y'' = -8e^{-2x} + c_1',$$
$$y' = 4e^{-2x} + c_1'x + c_2,$$

and

$$y = -2e^{-2x} + \tfrac{1}{2}c_1'x^2 + c_2x + c_3,$$

where c_1', c_2, and c_3 are arbitrary constants. If we replace the constant c_1' in the last formula by $2c_1$, it becomes

$$y = -2e^{-2x} + c_1x^2 + c_2x + c_3.$$

This last formula is slightly simpler in appearance. The two formulas describe the same set of functions since the coefficient of x^2 is completely arbitrary in both cases. Since $c_1' = 2c_1$, we see that to any arbitrarily assigned value for c_1, there corresponds a value for c_1' and vice versa.

If a formula can be found that describes the general solution of an nth-order equation, it usually involves n arbitrary constants. We note that this principle has been borne out in the last three examples, which admittedly are rather simple. Actually it is possible to find a simple formula that describes the general solution only for relatively few types of differential equations. Several such classes of first-order equations are discussed in the following three sections. In cases where it is not possible to find explicit formulas for the solutions, it still may be possible to discover certain properties of the solutions. For instance, it may be possible to show that a solution is bounded (or unbounded), to find its limiting value as the independent variable becomes infinite, or to establish that it is a periodic function. Much advanced work in differential equations is concerned with such matters.

Perhaps some reasons should now be given as to why we want to solve differential equations. Briefly, many experimentally discovered laws of science can be formulated as relations that involve not only magnitudes of quantities but also rates of change (usually with respect to time) of these magnitudes. Thus, the laws can be formulated as differential equations. A number of examples of problems that give rise to differential equations are presented in this book. Some applications will be described in Sections 1.5–1.9 after we have learned how to solve several kinds of first-order equations.

We have seen that ordinary differential equations can be classified as to order. We shall also categorize them in one more way. An equation of order n is said to be a *linear* equation if it is of the special form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x),$$

where $a_0, a_1, \ldots, a_n$ and f are given functions that are defined on an interval $\mathscr{I}$. Thus the general nth-order equation (1.4) is linear if the function F is a first-degree polynomial in $y, y', \ldots, y^{(n)}$. An equation that is not linear is said to be a *nonlinear* equation. For example, each of the equations

$$y' + (\cos x)y = e^x,$$
$$xy'' + y' = x^2,$$
$$xy''' - e^x y' + (\sin x)y = 0,$$

is linear, while each of the equations

$$y' + y^2 = 1,$$
$$y'' + (\cos x)yy' = \sin x,$$
$$y''' - x(y')^3 + y = 0,$$

is nonlinear. Because linear equations possess special properties, they will be treated in a separate chapter, Chapter 5.

In most applications that involve differential equations, the unknown function is required not only to satisfy the differential equation but also to satisfy certain other auxiliary conditions. These auxiliary conditions often specify the values of the function and some of its derivatives at one or more points. As an example, suppose we are asked to find a solution of the equation

$$\frac{dy}{dx} = 3x^2$$

that satisfies the auxiliary condition $y = 1$ when $x = 2$, or

$$y(2) = 1.$$

Thus, we require the graph of our solution (which is called a *solution curve* or *integral curve*) to pass through the point (2, 1) in the xy plane. The general solution of the equation is

$$y = x^3 + c,$$

where c is an arbitrary constant. In order to find a specific solution that satisfies the initial condition, we set $x = 2$ and $y = 1$ in the last formula, finding that $1 = 8 + c$ or $c = -7$. Thus, there is only one value of c for which the condition is satisfied. The equation possesses one and only one solution (defined for all x) that satisfies the condition, namely,

$$y = x^3 - 7.$$

For an nth-order equation of the form

$$y^{(n)} = G[x, y, y', y'', \ldots, y^{(n-1)}], \tag{1.5}$$

auxiliary conditions of the type

$$y(x_0) = k_0, \quad y'(x_0) = k_1, \quad y''(x_0) = k_2, \ldots, \quad y^{(n-1)}(x_0) = k_{n-1}, \tag{1.6}$$

where the k_i are given numbers, are common. We note that there are n conditions for the nth-order equation. These conditions specify the values of the unknown function and its first $n - 1$ derivatives at a single point x_0. For a first-order equation

$$y' = H(x, y),$$

we would have only one condition

$$y(x_0) = k_0$$

specifying the value of the unknown function itself at x_0. In the case of a second-order equation

$$y'' = K(x, y, y'),$$

we would have two conditions

$$y(x_0) = k_0, \qquad y'(x_0) = k_1.$$

A set of auxiliary conditions of the form (1.6) is called a set of *initial conditions* for the Eq. (1.5). The equation (1.5) together with the conditions (1.6) constitute an *initial value problem.* The reason for this terminology is that in many applications the independent variable x represents time and the conditions are specified at the instant x_0 at which some process begins.

In specifying the values of the first $n - 1$ derivatives of a solution of Eq. (1.5) at x_0, we have essentially specified the values of any higher derivatives that might exist. The values of these higher derivatives can be found from the differential equation itself. For example, let us consider the initial value problem

$$y'' = x^2 - y^3$$
$$y(1) = 2, \qquad y'(1) = -1.$$

From the differential equation we see that

$$y''(1) = 1 - 8 = -7.$$

By differentiating through in the differential equation, we find that

$$y''' = 2x - 3y^2y'$$

and hence

$$y'''(1) = 2 - (3)(4)(-1) = 14\,.$$

The values of higher derivatives at $x = 1$ can be found by repeated differentiation.

If a function can be expanded in a power series about a point x_0, a knowledge of the values of the function and its derivatves at x_0 completely determines the function. This discussion suggests that the initial value problem (1.5) and (1.6) can have but one solution if the function G is infinitely differentiable with respect to all variables. Actually it can be shown that, under rather mild restrictions on G, the initial value problem possesses a solution and that it has only one solution. In most of the problems and examples of this chapter, it is possible to actually find all the solutions of the differential equation at hand. In cases where this is impossible, it is comforting to know that the problem being considered actually has a solution and that there is only one solution. An initial value problem purporting to describe some physical process would not be very valuable without these two properties.

Exercises for Section 1.1

1. Find the order of the differential equation and determine whether it is linear or nonlinear.

(a) $y' = e^x$ (b) $y'' + xy = \sin x$
(c) $y' + e^y = 0$ (d) $y'' + 2y' + y = \cos x$
(e) $y'' + xyy' + y = 2$ (f) $y^{(4)} + 3(\cos x)y''' + y' = 0$
(g) $y''' = 0$ (h) $yy''' + y' = 0$

2. Find the general solution of the differential equation.

(a) $y' = 2x - 3$ (b) $y' = 3x^2 \sin x^3$
(c) $y' = \dfrac{4}{x(x-4)}$ (d) $y'' = 12e^{-2x} + 4$
(e) $y'' = \sec^2 x$ (f) $y'' = 8e^{-2x} + e^x$
(g) $y''' = 24x - 6$ (h) $y^{(4)} = 32 \sin 2x$

3. Find a solution of the differential equation that satisfies the specified conditions.

(a) $y' = 0, \quad y(2) = -5$
(b) $y' = x, \quad y(2) = 9$
(c) $y' = 4x - 3, \quad y(4) = 3$
(d) $y' = 3x^2 - 6x + 1, \quad y(-2) = 0$
(e) $y'' = 0, \quad y(2) = 1, \quad y'(2) = -1$
(f) $y'' = 9e^{-3x}, \quad y(0) = 1, \quad y'(0) = 2$
(g) $y'' = \cos x, \quad y(\pi) = 2, \quad y'(\pi) = 0$
(h) $y''' = e^{-x}, \quad y(0) = -1, \quad y'(0) = 1, \quad y''(0) = 3$

4. Show that a function is a solution of the equation $y' + ay = 0$, where a is a constant, if, and only if, it is a solution of the equation $(e^{ax}y)' = 0$. Hence show that the general solution of the equation is described by the formula $y = ce^{-ax}$, where c is an arbitrary constant.

5. Use the result of Exercise 4 to find the general solution of the given differential equation.

(a) $y' + 3y = 0$ (b) $y' - 3y = 0$
(c) $3y' - y = 0$ (d) $3y' + 2y = 0$

6. Verify that the differential equation has the given function as a solution.

(a) $xy' + y = 3x^2, \quad y = x^2,$ all x.
(b) $xy' + y = 0, \quad y = 1/x, \quad x > 0$.
(c) $y' + 2xy = 0, \quad y = \exp(-x^2),$ all x.
(d) $y'' + 4y = 0, \quad y = \cos 2x,$ all x.
(e) $y'' + y' - 2y = 0, \quad y = e^{-2x},$ all x.
(f) $2x^2y'' + 3xy' - y = 0, \quad y = \sqrt{x}, \quad x > 0$.

7. Verify that each of the functions $y = e^{-x}$ and $y = e^{3x}$ is a solution of the equation $y'' - 2y' - 3y = 0$ on any interval. Then show that $c_1e^{-x} + c_2e^{3x}$ is a solution for every choice of the constants c_1 and c_2.

8. Suppose that a function f is a solution of the initial value problem $y' = x^2 + y^2$, $y(1) = 2$. Find $f'(1)$, $f''(1)$, and $f'''(1)$.

9. If the function g is a solution of the initial value problem

$$y'' + yy' - x^3 = 0,$$

$$y(-1) = 1, \quad y'(-1) = 2,$$

find $g''(-1)$ and $g'''(-1)$.

10. Show that the problem $y' = 2x$, $y(0) = 0$, $y(1) = 100$, has no solution. Is this an initial value problem?